A Beautiful Geometric Property of the Complex Numbers: Statement and Proof in 6 Sentences

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Abstract

We relate the product of the vertices of a regular n-gon in the complex plane to the nth powers of the n-gon's center and complex radii.

Theorem: In the complex plane, consider a regular n-gon with center c and some complex radius r. Then the product of its vertices is equal to $c^n + r^n$ if n is odd, and $c^n - r^n$ if n is even. Symbolically: $\forall c, r \in \mathbb{C}, n \in \mathbb{N} = \{1, 2, ...\}$,

$$\prod_{k=1}^{n}(c + re^{2\pi ik/n}) = c^n - (-1)^n r^n \tag{1}$$

Proof: If the n-gon were centered at the origin, its vertices would be the nth roots of r^n, which are the zeros of $x^n - r^n$. So the vertices of our actual n-gon ('translated' to center c) are the zeros of $(x-c)^n - r^n$. Now, by Vieta's formulas, the product of the zeros of a monic polynomial of degree n is $(-1)^n$ times its constant term, and so the product of the n-gon's vertices is $(-1)^n((-c)^n - r^n)$, which equals $c^n + r^n$ if n is odd, and $c^n - r^n$ if n is even.

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