Unification of Quantum Mechanics and Gravitation Theory

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Abstract: Quantum mechanics and gravitation theory are unified here, when the unnormalized Schroedinger wave function of an isolated typical elementary particle, is given dimensions of potential energy, and is assumed to represent part of the particle's gravitational self potential energy, as well as representing the particle itself. This assumption is shown to be consistent with the normalization of the particle's wave function and its probabilistic interpretation. It leads directly to the derivation of a set of covariant partial differential equations which couple quantum mechanics, general relativity and electroweak and strong physics, and explains how particle rest mass arises quantum mechanically.The point of view taken here is that a necessary condition that a theory of elementary particle physics be fundamental, is that its defining equations be differential equations, exclusively. (For instance the electroweak field equations are the Bianchi identities satisfied by the gravitational field equations and lack parity invariant solutions). Gravitation theory is thereby inserted into quantum mechanics, and gravitation theory (including general relativity) and quantum mechanics are shown to be compatible theories.

Interacting particle systems of arbitrary complexity are represented here as the states occupied by a single gravitationally self interacting particle. If this single gravitationally self interacting particle is composed of matter and not antimatter, then a consequence of the theory is matter/antimatter asymmetry. Since the theory involves the gravitational self interaction of a single particle, any multiparticle system, can be represented in a background space which is ordinary four dimensional space time. This contrasts with conventional Schroedinger theory where the background space has 3N spatial dimensions and time, where N is the number of particles in the system. This simplification may lead to the numerical solution of complex molecular interactions, useful in drug, materials and energy research.

Solutions of the aforementioned covariant equations are shown to represent the following five categories of elementary particle states: 1. 3 spin ½ lepton states, 2. 3 spin ½ lepton states representing the antiparticles of the particles in category 1, 3. 6 spin ½ quarks, 4. 3 spin 1 bosons, and 5. 1 spin 0 boson. The strength of the gravitational self interaction at short range for leptons is shown to be proportional to $1/r^2$ and to be $1/r^4$ for quarks, where r is the distance from the particle.

The rotational stability of the galaxy based on elementary particle gravitational self interaction is shown.

Introduction

The Heisenberg Uncertainty Principal implies that the position of a single isolated particle with respect to its own gravitational field, is uncertain - the particle exists somewhere within its own gravitational field. Like any other particle whose motion would be affected by the isolated particle's gravitational field and whose energy would be quantized by it, the motion of the isolated particle that created the gravitational field, is assumed to be affected by its own gravitational field and its energy is assumed to be quantized by it. In the rest frame shared by the isolated particle and the time independent part of its own gravitational field, isolated particle energy is its rest energy and therefore particle rest energy and rest mass will be quantized by the isolated particle's own gravitational field. This reciprocity between the particle and its own gravitational field (the isolated particle creates its own gravitational field and the gravitational field affects the particle's motion and creates the particle by giving it rest mass) suggests that the particle's gravitational field and its Schroedinger wave function satisfy coupled partial differential equations.

There are at least three reasons why coupling the field equation of the particle's gravitational field and its Schroedinger equation should be impossible: A. The particle's gravitational field is a physical entity, and it is not certain that the complex unnormalized Schroedinger wave function, which could lack physical dimensions and which might not be physically real, could be coupled to it, B. Solutions of the Schroedinger equation normally lack a time independent part, and it is uncertain that the Schroedinger equation could be coupled to the field equation of the particle's gravitational field, whose solution has a time independent part, and C. Coupling the particle's Schroedinger equation to the differential equation of the particle's gravitational field could make wave function normalization and the probability interpretation of the Schroedinger wave function, impossible.

Condition A. above is addressed by assuming that the unnormalized Schroedinger wave function of an isolated particle represents part of the particle's own gravitational self potential energy field and has dimensions of potential energy. This makes the wave function physically real and possibly measurable The wave function is complex however, and would represent a complex gravitational field. This might permit the modeling of particle creation and annihilation during collisions, just as a complex refractive index allows a light beam to gain or lose energy (create or annihilate photons) as it traverses a medium. If the Schroedinger wave function represents gravitational self potential energy, then gravitation theory is thereby inserted into quantum mechanics, by giving the wave function dimensions of potential energy. This simple procedure unites gravitation theory and Schroedinger quantum mechanics and is unaffected by any nonlinearity of the gravitational field equations or by the possible nonexistence of the graviton. This simplicity contrasts with the complex conventional approach for uniting gravitation and quantum mechanical theory, whereby quantum field theory is inserted into gravitation theory. This latter procedure has never worked and is probably adversely affected by gravitational field nonlinearity and the absence of the graviton. It is shown shortly that conditions B. and C. above are also addressed by endowing the wave function with dimensions of potential energy.

What are the physical dimensions of the unnormalized Schroedinger wave function? Since the wave function occurs to first power in each term of the Schroedinger equation, its dimensions are undefined by the Schroedinger equation. In addition the normalized wave function always has dimensions of one over the square root of volume, regardless of the dimensions of the unnormalized wave function. Therefore, motivated by the desire to give the wave function a physical basis, to unify gravitation theory and quantum mechanics, and to produce a quantum mechanical theory of elementary particle rest mass, the wave function is assumed to have dimensions of potential energy.

If the wave function Ψ is assigned dimensions of potential energy, it can be added to or subtracted from the potential energy function V in the Schroedinger equation, where

V and Ψ represent two different parts of the particle's gravitational self potential energy, to form two new potential energy functions, that are dimensionally consistent. The two potential energy functions are called U_{sr1} and U_{sr2}, where $U_{sr1} = V + \Psi$ and $U_{sr2} = V - \Psi$ and it can be shown that these potential energy functions are consistent with the following partial differential equation

$$ih\partial U/\partial t = -h^2/2m_0 \nabla^2 U + m_0c^2 U + U^2/2 \tag{1}$$

where h is Planck's constant divided by 2π and U is called the modified wave function and has dimensions of potential energy. If U_{sr1} as defined above is an assumed solution of (1) and is first inserted into (1), and then U_{sr2} as defined above is an assumed solution of (1) is then inserted into (1), and then the two differential equations for U_{sr1} and U_{sr2} are subtracted, the Schroedinger equation for Ψ is recovered. Adding the two partial differential equations, produces a coupled partial differential equation for the quantizing potential energy function V. Therefore (1) can be decomposed into

$$ih\partial V/\partial t = (-h^2/2m_0)\nabla^2 V + m_0c^2 V + V^2/2 + \Psi^2/2 \tag{2}$$

$$ih\partial \Psi/\partial t = (-h^2/2m_0)\nabla^2 \Psi + m_0c^2 \Psi + V\Psi \tag{3}$$

If V represents part of the particle's quantizing field of gravitational self potential energy, then (2) and (3) represent the self interaction of a particle with its own quantizing field. Since (2) and (3) can be combined into (1), it is assumed that V, Ψ and U share the same rest frame. Since V is gravitational and attractive, the particle will occupy bound states with quantized energy. In the rest frame shared by the particle and its quantizing field, the particle's energy is its rest energy, and particle rest energy and rest mass will be quantized, endowing the particle with rest mass. Therefore the particle's interaction with its own quantizing field, may be the reason it has rest mass. (In this paper it is assumed that an elementary particle is a bound state of its own gravitational self potential energy field and that in the rest frame shared by the particle and its own gravitational self potential energy field, that particle rest energy and rest mass are quantized)

Inserting the solution $U = \Sigma_n U_n e^{-inEt/h}$, n=0,1,2,3,4,……, into (1) or $V = \Sigma_n U_n e^{-inEt/h}$, n= 0,2,4,6,8, ……, and $\Psi = \Sigma_n U_n e^{-inEt/h}$, n=1,3,5,7,……, into (2) and (3) and grouping terms with the same time exponential, leads to an infinite sequence of coupled partial differential equations of which the following are the first two equations in the sequence

$$0 = (-h^2/2m_0)\nabla^2 U_0 + m_0c^2 U_0 + U_0^2/2 \tag{4}$$

$$EU_1 = (-h^2/2m_0)\nabla^2 U_1 + m_0c^2 U_1 + U_0 U_1 \tag{5}$$

These solutions for U, V, Ψ show that a solution for V containing a time independent part could coexist with a solution for Ψ which lacks a time independent part and resolves condition B above. Also (5) is the time independent Schroedinger equation for U_1 and for appropriate U_0 could lead to normalizable solutions for U_1 . In addition (5) can be associated with a time dependent Schroedinger equation, therefore the normalization condition and the probability interpretation applies to at least a part of U and ameliorates condition C above.

There is a problem with equation (1). The potential energy function V is normally associated with particles which are bosons, and the wave function Ψ is normally associated with particles which are Fermions. In addition there are two ways to define the joint wave function of two particles 1 and 2 with individual wave functions Ψ_1 and Ψ_2 in order that the joint wave functions of the particles have dimensions of potential energy. These joint wave functions are $\Psi_1 + \Psi_2$ and $\Psi_1 - \Psi_2$. If the two particles are identical and they occupy the same state, the second joint wave function goes to zero (indicative of Fermions) and the first joint wave function representing the increased probability that the two identical particles occupy the same state is indicative of the behavior of bosons.

How can the particle whose field equation is (1), be both a Fermion and a Boson at the same time? If there is only one such gravitationally self interacting particle in any closed system, such as the universe, the problem is resolved. The question of whether the particle is a Fermion or a Boson is indeterminate, and it is assumed that the particle can occupy particle states representing Fermions or Bosons and endowing them with rest mass. If this single gravitationally self interacting particle is composed of either matter or antimatter, but not both simultaneously , then a consequence of this theory is matter/antimatter asymmetry. This coincides with the observation that our universe is dominated by the presence of matter.

In addition, since there is only one such particle in the closed system and there could be many particle states which exist simultaneously, it is assumed that solutions of (1) include multiparticle states of Fermions and Bosons. Although (1) is the field equation of a single particle, its solutions can represent extremely complex systems of particle states. Therefore (1) may represent the evolution of a multiparticle system as the gravitational self interaction of a single particle. The solution above that led to (4) and (5) can be extended to systems with any number of particles. For instance the solution of (1) for the two particle case is $U = \Sigma_{m,n} U_{mn} \exp(mE_1 + nE_2)$, m,n = 0,1,2,3.... Therefore (1) is a single particle equation with multiparticle solutions. In a later section of this paper, where the covariant tensor form of (1) is solved, it is shown that this equation has solutions that can represent Fermions and solutions that can represent Bosons. Did the particle obeying (1) occupy all the particle states present at the birth of the universe and endow them all with rest mass at that time, or does it continue to occupy all the particle states in the universe, endowing them with rest mass and somehow communicating with them all?

Since (5) is the time independent Schroedinger equation with potential energy function U_0 and can be associated with the time dependent Schroedinger equation, these equations show that (4) with solutions possessing a time independent part can coexist with the time dependent Schroedinger equation, thereby resolving condition B. above. If U_0 is real and attractive, solution of (5) could lead to normalizable wave functions U_1 and a probabilistic interpretation of U_1 , the first term in the solution for Ψ. Therefore the initial assumption that Ψ could represent part of a particle's gravitational self potential energy function appears to be compatible with normalization of the wave function and a probability interpretation of Ψ, resolving condition C. above. In the particle's rest frame $E = m_0c^2$ and particle rest energy quantization implies rest mass quantization. If U_0 is attractive, (5) could lead to bound state functions U_1 which are normalizable. The simultaneous solution of (4) and (5) with $E = m_0c^2$ might lead to a discrete set of values for m_0.

If U represents a particle's gravitational self potential energy, it should be possible to relate it to φ, the particle's classical gravitational potential. It should also be possible to obtain the field equation for the particle's entire nonrelativistic gravitational field. Since V can have a time independent real part, U can have a time independent real part and (1) should have a real, time independent solution. It will now be shown that (1) can be used to relate U to φ. The relativistically covariant form of the resulting gravitational field equation and the relativistically covariant equivalent of (1) will later be shown to merge quantum mechanics and general relativity into a four dimensional, background independent theory which implies the existence of the particle's electroweak field.

The classical gravitational field equation of a point particle with gravitational rest mass m_0 is

$$\nabla^2 m_0\varphi = 0 \qquad (6)$$

Where $\varphi = -km_0/R$ is the Newtonian gravitational potential and $m_0\varphi$ has dimensions of potential energy. Equation (6) is the differential equation of the particle's gravitational field in a nonrelativistic, linearized, time independent theory of gravity. To combine (1) and (6) consistently, the linearized time independent version of nonrelativistic equation (1) must be used and it is

$$\nabla^2 U_0 = (2m_0^2c^2/h^2)U_0 \qquad (7)$$

where U_0 is the time independent part of U. Consider the following gravitational field equation

$$\nabla^2 m_0\varphi = (2m_0^2c^2/h^2)U_0 \qquad (8)$$

where U_0 is the solution of (7). Equation (8) has a homogeneous solution given by the solution of (6), and an inhomgeneous solution given by the solution of (7). In other words the equation satisfied by the inhomogeneous solution of (8) is the same as the linearized time independent version of (1).Since (8) is linear, a complete solution of (8) is given by

$$m_0\varphi = -km_0^2/R + U_0 = -km_0^2/R + Ae(-\sqrt{2}\alpha R)/R \qquad (9)$$

where $\alpha = m_0c/h$. If $A = km_0^2$, the potential energy function $m_0\varphi$ in (9) and its gradient are nonsingular for $\text{Lim } R \to 0$. If the particle's total gravitational field energy is defined as $1/8\pi k$ times the volume integral of the square of the gradient of φ, then the energy in the point particle's gravitational field given by (9) is finite. In conventional theory the energy in the gravitational field of a point particle of finite mass is infinite. Given the equivalence of mass and energy this would lead to the contradictory conclusion that the particle's rest mass is infinite. Also as will later be shown, the relativistic covariant equivalents of (1) and (8) along with associated Bianchi identities can be derived. These are a set of coupled partial differential equations for the metric tensor, the modified wave function tensor and the stress energy tensor of the electroweak field. These equations may unify quantum mechanics, General Relativity and electroweak physics in a four dimensional background independent theory. Equation (1) represents the gravitational self potential energy field of an isolated particle and (8) represents the isolated particle's gravitational field.

The classical equation of motion of an isolated point, elementary particle that interacts with its own gravitational field, and whose gravitational field obeys (8) and (9) is given by

$$m_0 dv/dt = - \text{grad}(m_0\varphi) \tag{10}$$

Where v is the particle's classical velocity vector and $-\text{grad}(m_0\varphi)$ is the particle's gravitational self force, which from (9) points toward the center of the particle's own gravitational self field. Equations (9) and (10) can be used to calculate the gravitational self force on an isolated particle

Newton's laws of motion involve external forces acting on a particle. In particular, Newton's first law of motion states that if the net external force acting on a particle is zero, that the particle's motion is unaccelerated. The particle whose average motion obeys (10) is not acted on by any external forces, since the gravitational self force is internally generated, but the particle's average motion is accelerated. The particle obeying (10) has an average motion that would violate Newton's first law of motion. How would the motion of a composite body, like a star, composed of quarks and gluons be affected by the gravitational self interaction force of the quarks and gluons? In the next section, it is shown, that the rotational stability of the galaxy might be due to quark and gluon gravitational self interaction, and not to the presence of dark matter.

Gravitational Self Force and Particle Macroscopic Motion

A star's mass is composed primarily of protons, and the protons are in turn composed of quarks and gluons. In the following, the microscopic motion of protons, quarks and gluons within stars is ignored, and the motion of these particles is assumed to be the same as the macroscopic orbital motion of the stars they occupy about the galactic center. Also, equations (9) and (10) cannot be applied directly to the calculation of the gravitational self force on a quark or gluon in a proton, since the quark or gluon move at relativistic speeds within the proton, and (9) and (10) were derived for particles moving at nonrelativistic speeds. It is assumed that the gravitational self force on a relativistic quark or gluon is the same as the gravitational self force acting on an equivalent particle at rest within the proton, whose mass is equal to the part of the proton's rest mass contributed by the relativistic quark or gluon. Also if the gravitational self force acting on an equivalent particle is initially perpendicular to the star's velocity vector and points toward the galactic center (as well as to the center of the particle's own gravitational field), it will continue to do so thereafter, because the equivalent particle's travel path is slightly longer than the path traveled by its gravitational center. This is so because the equivalent particle's gravitational field strength is greater at the center of its gravitational field, than at the particle location in its own gravitational field, and the warpage or shortening of space is greater at the gravitational field's center than at the particle location. Therefore the particle and its gravitational field center could move along concentric circles, centered at the galactic center, with the particle having a slightly larger radius than its gravitational center. It is now shown that this is probably true and can explain the motion of stars in galaxies.

It is further assumed that half the proton's rest mass is attributable to the three quarks of which it is composed and half the proton's rest mass is attributable to the eight gluons of which it is composed. The proton rest mass is 1.67×10^{-24} grams, the part of this mass contributed by each moving quark is about one sixth of this or 2.78×10^{-25} grams and the part contributed by each gluon is about one sixteenth of the proton rest mass or 1.04×10^{-25} grams. The gravitational self force on each equivalent particle of a relativistic quark and gluon is calculated, using the gradient of $m_0\varphi$ in (9) with m_0 either 2.78×10^{-25} for a quark or 1.04×10^{-25} for a gluon. Then the gravitational self force on the proton can be calculated by adding the contributions of each equivalent of a quark or gluon. Assuming that each quark and gluon is at the center of its gravitational self field ($R = 0$) in the expression for grad($m_0\varphi$), where the self force is maximized, the proton self force is found to be 1.01×10^{-30} dynes. This force declines as the distance of each quark and gluon from its own center of gravitation increases.

Examination of the rotation curve of our galaxy (see Astronomy, the Evolving Universe, Cambridge University Press, 2002 by Michael Zeilik, P.395, figure 17.8) shows the velocity of stars whose distance from the galactic center varies from 1000 light years to 50000 light years to be roughly constant and equal to 250km/sec. The centrifugal force on protons m_0v^2/R_g moving at this speed in circular orbits about the galactic center, where R_g is the proton's radial distance from the galactic center, with R_g ranging from 1000 to 50000 light years, varies from 1.10×10^{-30} dynes to 2.2×10^{-32} dynes. The small discrepancy between this result and the result for the maximum gravitational self force might be due to the gravitational effect of other stars in the galaxy. The contribution of the gravitational self force to the rotational stability of the galaxy is therefore about 90%. Given the approximate nature of the assumptions made in this analysis, the magnitude of the calculated gravitational self force on a proton seems to be large enough to produce the observed galactic rotation curve and in remarkable agreement with observation. It is also found that 1000 light years is the smallest radius at which protons and stars can be held in orbit by the gravitational self force calculated using (9) and moving at 250 km per second in our galaxy. In agreement with figure 17.8 referenced above.

In addition, the self force on a proton in orbit about the sun, at the same distance from the sun as the earth, can be calculated and is found to be between five and six orders of magnitude smaller than the gravitational force exerted by the sun on the proton. Therefore, at the level of solar system distances, the gravitational self force can be ignored, compared to the usual gravitational force between bodies.

Therefore the rotational stability of our galaxy could conceivably be explained by the gravitational self force on a proton rather than the assumed existence of dark matter.

Derivation of the Covariant Equations of Elementary Particle Gravitational Self Interaction

In this paper it is assumed that the metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ in flat space and rectangular coordinates are given by

$$g_{\alpha\beta} = \begin{matrix} -1/c^2 & 0 & 0 & 0 \\ 0 & -1/c^2 & 0 & 0 \\ 0 & 0 & -1/c^2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad g^{\alpha\beta} = \begin{matrix} -c^2 & 0 & 0 & 0 \\ 0 & -c^2 & 0 & 0 \\ 0 & 0 & -c^2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad (11)$$

Assume that the relativistic generalization of (1) has the following form

$$h^2\Box^2 U = - m_0^2 c^4 U - \bar{\alpha} U^2 \qquad (12)$$

Where $\Box^2 = \partial^2/\partial t^2 - c^2\nabla^2$ and $\bar{\alpha}$ is a constant to be determined. (12) is the relativistic wave equation of a spin zero free particle modified by the quadratic term in U to account for particle gravitational self interaction. Inserting U_{sr1} and U_{sr2} defined previously into (12) and subtracting the resulting two equations gives

$$h^2\Box^2\Psi = - m_0^2 c^4\Psi - 2\bar{\alpha} V\Psi \qquad (13)$$

The nonquantum mechanical relativistic energy equation associated with (13) is

$$E^2 = m_0^2 c^4 + c^2 p^2 + 2\bar{\alpha} V \qquad (14)$$

The classical nonrelativistic energy equation associated with (1) is

$$E = m_0 c^2 + p^2/(2m_0) + V \qquad (15)$$

In the particle's rest frame $E = m_0c^2$ and in the particle's rest frame (14) and (15) should become identical. This happens if $\bar{\alpha} = m_0 c^2$ and then (12) becomes

$$\Box^2 U = -(m_0^2 c^4/h^2)U - (m_0 c^2/h^2)U^2 \qquad (16)$$

Equation (16) is the relativistic generalization of (1). Also if U is a solution of (16), then so is U^*, the complex conjugate of U, and it is assumed that U^* represents the antiparticle of the particle represented by U.

Since it has been shown that U could represent a kind of short range gravitational self potential energy field, it is assumed that U like the metric tensor is a second rank tensor U_{ij} . Therefore (16) is replaced by the following relativistically covariant equation:

$$g^{\alpha\beta}U^i_{j;\alpha\beta} = -(m_0^2c^4/h^2)U^i_j - (m_0c^2/h^2)U^i_\alpha U^\alpha_j \qquad (17)$$

where g_{ij} is the metric tensor, ; represents covariant differentiation and $i,j,\alpha,\beta = 1,2,3,4$, sum on α and β. Equation (17) possesses the solution $U^i_j = f_0\delta^i_j$ where δ is the Kronecker tensor. It can be shown by substitution into (17) that f_0 satisfies (16), which is the modified wave equation of a structureless spin zero boson, interacting with its own gravitational field .

To make the theory developed here background independent and to show that U_{ij} represents gravitational self potential energy, a covariant gravitational field equation for the metric tensor which is coupled to (17) is derived. Tentatively consider the following gravitational field equation which is the covariant generalization of (8)

$$R_{ij} = \bar{a}U_{Sij} - (2K/c^6)\Phi_{ij} \quad i,j = 1,2,3,4 \qquad (18)$$

where R_{ij} is the second rank curvature tensor and is symmetric in the subscripts i, j , $\bar{a}$ is a constant to be determined, U_{Sij} and U_{Aij} are the parts of U_{ij} which are respectively symmetric and antisymmetric in the subscripts i,j and Φ_{ij} is a symmetric second rank tensor which will later be shown to contain the electroweak stress energy tensor (see appendix 1). Φ_{ij} allows (18) to satisfy Bianchi identities and its coefficient has been chosen for later convenience. If the particle represented by (18) is electrically neutral and U_{Sij} declines quickly enough as $r\rightarrow\infty$, where r is the distance from the particle, then $R_{ij} = 0$ far from the particle and the empty space gravitational field equation is recovered.

When (18) is linearized, its (4,4) component equation will yield a differential equation analogous to (8) for the scalar gravitational self potential energy function $m_0\varphi$, where φ is the scalar gravitational potential. In order that this equation give a finite value for the energy in the gravitational field of a mass point, the solution of the differential equation satisfied by $m_0\varphi$ must consist of the sum of two solutions (as in (8)): 1. a solution which is the Newtonian mass point gravitational potential function multiplied by m_0 and (2) a solution for $m_0\varphi$ which is analagous to (8) and equals U_{S44} . This condition makes it possible to determine $\bar{a}$.

Using the textbook definition of R_{ij} in terms of Christoffel symbols and the textbook definition of the Christoffel symbols in terms of the metric tensor g_{ij} , it can be shown that in the time independent,weak field case that

$$R_{44} = -\tfrac{1}{2} c^2\nabla^2 h_{44} = \bar{a}U_{S44} - (2K/c^6)\Phi_{44} \qquad (19)$$

$$\text{where } g_{44} = 1 + h_{44} \qquad (20)$$

(See Gravitation and Cosmology, Principals and Application of the General Theory of Relativity, 1st addition, 1972, see the equation before 10.1.4 and equation 10.1.1)

For the case of a single isolated particle without gravitational self interaction, the right side of (19) equals zero and

$$R_{44} = 0 \qquad (21)$$

and the Schwartzchild solution for g_{44} and h_{44} applies giving

$$\tfrac{1}{2}\, m_0c^2h_{44} = \tfrac{1}{2}\, m_0c^2\bullet(-2Km_0/(c^2R)) = -Km_0^2/R = (m_0\varphi) \qquad (22)$$

where φ is the Newtonian gravitational potential of a point mass. It is further assumed that this definition between h_{44} and φ holds when the right side of (19) is not zero, giving

$$\nabla^2(m_0\varphi) = -\,\bar{a}m_0\, U_{S44} + (2Km_0/c^6)\Phi_{44} \qquad (23)$$

The relationship between $m_0\varphi$ in (23) and U_{s44} is now established. Equation(17) may be decomposed into two parts which are symmetric and antisymmetric in the subscripts i and j, where U_{Sij} and U_{Aij} are given by

$$U_{Sij}=\tfrac{1}{2}(U_{ij}+ U_{ji}),\quad U_{Aij} = \tfrac{1}{2}\,(U_{ij} - U_{ji}) \qquad (24)$$

Inserting these into (17) and separating (17) into equations which are symmetric and antisymmetric in the subscripts i and j , gives

$$g^{\alpha\beta}U_{Sij;\alpha\beta} =-(m_0^2c^4/h^2)U_{Sij} - (m_0c^2/h^2)(U_{Si\alpha}U_S{}^{\alpha}{}_j + U_{Ai\alpha}U_A{}^{\alpha}{}_j) \qquad (25)$$

$$g^{\alpha\beta}U_{Aij;\alpha\beta} = -(m_0^2c^4/h^2)U_{Aij} - (m_0c^2/h^2)(\, U_{Ai\alpha}\, U_S{}^{\alpha}{}_j + U_{Si\alpha}\, U_A{}^{\alpha}{}_j) \qquad (26)$$

In the linearized time independent case the (4,4) component of (25) in rectangular coordinates becomes

$$\nabla^2U_{S44} = (m_0^2c^2/h^2)U_{S44} \qquad (27)$$

If U_{s44} represents gravitational self potential energy, it should be possible to relate it to the particle's gravitational potential φ as was done in (8) in the first section. The solution of (23) consists of the sum of a homogeneous solution, where the right side of (23) equals zero and an inhomogeneous solution where the solution of (23) equals U_{s44}. If $\bar{a}= -m_0c^2/h^2$, (23) becomes

$$\nabla^2(m_0\varphi) = (m_0^2c^2/h^2)U_{s44} + (2Km_0/c^6)\Phi_{44} \qquad (28)$$

and if (27) is modified to include Φ_{44} it becomes

$$\nabla^2U_{S44} = (m_0^2c^2/h^2)U_{S44} + (2Km_0/c^6)\Phi_{44} \qquad (29)$$

and the inhomogeneous solution of (28) satisfies (29) with $m_0\varphi$ equal to U_{s44}.

Inserting the above value for $\bar{a}$ into (18) gives

$$R_{ij} = -(m_0c^2/h^2)U_{Sij} - (2K/c^6)\Phi_{ij} \qquad (30)$$

and using (29) equation (17) becomes

$$g^{\alpha\beta}\, U_{ij;\alpha\beta} = -(m_0^2c^4/h^2)U_{ij} - (m_0c^2/h^2)U_{i\alpha}\, U^{\alpha}{}_j - (2Km_0/c^4)\Phi_{ij} \tag{31}$$

Equation (30) may be rewritten as

$$R^i{}_j\text{-}½R\delta^i{}_j = -(m_0c^2/h^2)(U_S{}^i{}_j\text{-}½\, U_S{}^{\alpha}{}_{\alpha}\delta^i{}_j) - (2K/c^6)(\Phi^i{}_j\text{-}½\, \Phi^{\alpha}{}_{\alpha}\delta^i{}_j) \tag{32}$$

where $R = R^{\alpha}{}_{\alpha}$, sum on α. and $\delta^i{}_j$ is the Kronecker tensor.
Inserting the following into (32)

$$U_S{}^i{}_j = f_0\delta^i{}_j + H_S{}^i{}_j, \quad U_A{}^i{}_j = H_A{}^i{}_j \tag{33}$$

($H_S{}^i{}_j$ and $H_A{}^i{}_j$ are traceless) , taking the covariant derivitive of (32) and using the Bianchi identity

$$(R^i{}_j\text{-}½R\delta^i{}_j)_{;i} = 0 \tag{34}$$

gives

$$(2K/c^6)(\Phi^i{}_j\text{-}½\Phi^{\alpha}{}_{\alpha}\delta^i{}_j)_{;i} = (m_0c^2/h^2)f_{0,j} - (m_0c^2/h^2)H_S{}^i{}_{j;i} \tag{35}$$

Since (35) is linear in $\Phi^i{}_j$ it can be split into the sum of three parts

$$\Phi^i{}_j = \Phi_1{}^i{}_j + \Phi_2{}^i{}_j + \Phi_3{}^i{}_j \quad (\Phi_1{}^i{}_j \text{ and } \Phi_2{}^i{}_j \text{ are traceless}) \tag{36}$$

and $$\Phi_3{}^i{}_j = .25\Phi_3{}^{\alpha}{}_{\alpha}\delta^i{}_j \tag{37}$$

Then $$(2K/c^6)\Phi_1{}^i{}_{j;i} = (m_0c^2/h^2)f_{01,j} \tag{38}$$

$$(2K/c^6)\Phi_2{}^i{}_j = -(m_0c^2/h^2)H_S{}^i{}_j \tag{39}$$

$$(K/2c^6)\Phi_3{}^{\alpha}{}_{\alpha} = -(m_0c^2/h^2)f_{02} \tag{40}$$

where $$f_0 = f_{01} + f_{02} \tag{41}$$

and f_{01} is composed of the sum of terms, each declining exponentially with distance and possessing a singular part $1/R^N$, N=1,2 and f_{02} is composed of the sum of terms, each of which either lack the exponential part or possess a singularity of order equal to or higher than 3.

The covariant derivitive is absent from (39) since both sides of the equation are traceless and the covariant differential equation is satisfied if both sides are equal. The covariant derivitive is absent from (40), since both sides of the equation are scalars and the covariant differential equation is satisfied if both sides are equal. $\Phi_1{}^i{}_j$ must be traceless or (35) and (38) could be replaced by equalities without derivatives. Then the right side of (32) would equal zero and (32) would reduce to $R^i_j = 0$. This would lead to $\nabla^2 \varphi = 0$ in the linearized case, and the prediction that the total gravitational field energy of a mass point would be infinite - an impossibility.

Inserting (33) and (36) into (31) gives

$$g^{\alpha\beta} f_{0;\alpha\beta} = -(m_0^2 c^4/h^2) f_0 - (m_0 c^2/h^2)(f_0^2 + H_S{}^\beta{}_\alpha H_S{}^\alpha{}_\beta/4 + H_A{}^\beta{}_\alpha H_A{}^\alpha{}_\beta/4) + (m_0^2 c^4/h^2) f_{02} \quad (42)$$

$$g^{\alpha\beta} H_A{}^i{}_{j;\alpha\beta} = -(m_0^2 c^4/h^2) H_A{}^i{}_j - (m_0 c^2/h^2)(2 f_0 H_A{}^i{}_j + H_A{}^i{}_\alpha H_S{}^\alpha{}_j + H_S{}^i{}_\alpha H_A{}^\alpha{}_j) \quad (43)$$

$$g^{\alpha\beta} H_S{}^i{}_{j;\alpha\beta} = -(m_0^2 c^4/h^2) H_S{}^i{}_j - (m_0 c^2/h^2)[2 f_0 H_S{}^i{}_j + H_S{}^i{}_\alpha H_S{}^\alpha{}_j - (1/4) H_S{}^\beta{}_a H_S{}^\alpha{}_\beta \delta^i_j +$$

$$+ H_A{}^i{}_\alpha H_A{}^\alpha{}_j - (1/4) H_A{}^\beta{}_\alpha H_A{}^\alpha{}_\beta \delta^i_j] - (2Km_0/c^4)(\Phi_1{}^i{}_j + \Phi_2{}^i{}_j) \quad (44)$$

Inserting (39) into (44) gives

$$g^{\alpha\beta} H_S{}^i{}_{j;\alpha\beta} = -(m_0 c^2/h^2)[2 f_0 H_S{}^i{}_j + H_S{}^i{}_\alpha H_S{}^\alpha{}_j - (1/4) H_S{}^\beta{}_a H_S{}^\alpha{}_\beta \delta^i_j$$

$$+ H_A{}^i{}_\alpha H_A{}^\alpha{}_j - (1/4) H_A{}^\beta{}_\alpha H_A{}^\alpha{}_\beta \delta^i_j] - (2Km_0/c^4)\Phi_1{}^i{}_j \quad (45)$$

Inserting (33), (36) and (39) into (32) gives

$$R^i_j - \tfrac{1}{2} R \delta^i_j = (m_0 c^2/h^2) f_{01} \delta^i j - (2K/c^6)\Phi_1{}^i{}_j \quad (46)$$

The field equations of a gravitationally self interacting mass point are summarized here:

$$R^i_j - \tfrac{1}{2} R \delta^i_j = (m_0 c^2/h^2) f_{01} \delta^i_j - (2K/c^6)\Phi_1{}^i{}_j \quad (47)$$

$$(2K/c^6)\Phi_1{}^i{}_{j\,;i} = (m_0 c^2/h^2) f_{01,j} \quad (48)$$

$$g^{\alpha\beta} f_{0;\alpha\beta} = -(m_0^2 c^4/h^2) f_0 - (m_0 c^2/h^2)(f_0^2 + H_S{}^\beta{}_\alpha H_S{}^\alpha{}_\beta/4 + H_A{}^\beta{}_\alpha H_A{}^\alpha{}_\beta/4) + (m_0^2 c^4/h^2) f_{02} \quad (49)$$

$$g^{\alpha\beta} H_A{}^i{}_{j;\alpha\beta} = -(m_0^2 c^4/h^2) H_A{}^i{}_j - (m_0 c^2/h^2)(2 f_0 H_A{}^i{}_j + H_A{}^i{}_\alpha H_S{}^\alpha{}_j + H_S{}^i{}_\alpha H_A{}^\alpha{}_j) \quad (50)$$

$$g^{\alpha\beta} H_S{}^i{}_{j;\alpha\beta} = -(m_0 c^2/h^2)[2 f_0 H_S{}^i{}_j + H_S{}^i{}_\alpha H_S{}^\alpha{}_j - (1/4) H_S{}^\beta{}_a H_S{}^\alpha{}_\beta \delta^i_j$$

$$+ H_A{}^i{}_\alpha H_A{}^\alpha{}_j - (1/4) H_A{}^\beta{}_\alpha H_A{}^\alpha{}_\beta \delta^i_j] - (2Km_0/c^4)\Phi_1{}^i{}_j \quad (51)$$

$$f_0 = f_{01} + f_{02} \quad (52)$$

where f_{01} is composed of the sum of terms each declining exponentially with distance and possessing a singular part $1/R^N$, N= 1,2 and f_{02} is composed of the sum of terms each of which either lack the exponential part or possess a singularity of order equal to or greater

than 3 and $i,j,\alpha\beta=1,2,3,4$, sum on α,β.
(Note: In rectangular coordinates in the flat space approximation, (48) is a first order differential equation. and the transformation $x_i \rightarrow -x_i$, $i=1,2,3$ changes the sign of the terms in (48) involving $\partial/\partial x_i$, $i = 1,2,3$ and not any other term. Therefore the parity transform of (48) is a different equation than (48). The parity transform of a solution of (48), Φ^i_j (-x,-y,-z,t) satisfies the parity transform of (48) which is different from (48). Since Φ^i_j(-x,-y,-z,t) satisfies a different equation than (48), it can't generally be equal to $\pm\Phi$(x,y,z,t), which is a solution of (48), and parity is not conserved by the solutions of (48).)

It is shown in standard texts that if $\Phi_1{}^i_j$ is the stress energy tensor of the electromagnetic field when $f_{01}=0$, that (48) reduces to Maxwell's electromagnetic field equations in empty space (See "General Theory of Relativity" by P.A.M. Dirac Copyright 1975, pages 54 to 58) . It is also shown in standard texts on classical general relativity that the second term on the right side of (47) gives the effect of the electromagnetic field on the gravitational field of a particle if $\Phi_1{}^i_j$ is the stress energy tensor of the electromagnetic field and where

$$\Phi_1{}^i_j = (\varphi^i{}_\alpha\varphi^\alpha{}_j - \tfrac{1}{4}\,\varphi^\alpha{}_\beta\varphi^\beta{}_\alpha\delta^i_j)/(4\pi c^6) \qquad (53)$$

where φ^i_j is the electromagnetic field tensor.

The covariant equivalent of (8) is

$$d\,\bar{U}^\mu/d\tau+\{^\mu{}_{\rho\sigma}\}\,\bar{U}^\rho\,\bar{U}^\sigma-(e/(m_0c^3))\varphi^{\mu\rho}\,\bar{U}_\rho=0 \qquad (54)$$

where $\bar{U}^\mu$ is the particle's velocity four vector, $d\tau$ is the proper time and e is the electronic charge.

When $f_{01} \neq 0$, (48) does not reduce to Maxwell s equations. (See appendix 1 where $\Phi_1{}^i_j$ in flat space is expressed in terms of electric and magnetic components in polar coordinates and appendix 2 where (48) in flat space is expressed in polar coordinates) If $f_{01} \neq 0$ only in the immediate vicinity of the particle, then far from the particle, $\Phi_1{}^i_j$ obeys Maxwell s equations and is smooth and continuous, and in the immediate vicinity of the particle, $\Phi_1{}^i_j$ does not obey Maxwell s equations. It can also be shown for an isolated particle that (48) has solutions when $f_{01} \neq 0$ which completely lack a long range electromagnetic field, and the particle represented is uncharged and lacks a dipole moment. $\Phi_1{}^i_j$ in this case may represent the particle's electroweak field and (48) is the field equation of the particle's electroweak field.

Solution of the Covariant Quantum Mechanical Field Equation and the Representation of Elementary Particle States

The solutions for $H_A{}^i{}_j$ and $H_S{}^i{}_j$ in (49), (50), (51) in flat space time and rectangular coordinates are given by

$$H_A{}^i{}_j = \Sigma_N f_N H_N{}^i{}_j \text{ ,N= 1,2, ..6 and } H_S{}^i{}_j = \Sigma_N f_N H_N{}^i{}_j \text{ ,N = 7,8,9, ...,15} \qquad (55)$$

where f_N , N = 1,2, ...15 are 15 scalar functions of space and time and $H_N{}^i{}_j$, N= 0,1,2.....15 are 16 mixed contravariant covariant second rank tensors including the identity matrix. (The effect of f_{02} and $\Phi_1{}^i{}_j$ on (49) and (51) is ignored here for simplicity). These matrices include H_0 = I the identity matrix, and the other matrices are traceless and given by the following

$$H_1 = \begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{matrix} \qquad H_2 = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{matrix} \qquad H_3 = \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

$$H_4 = \begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{matrix} \qquad H_5 = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{matrix} \qquad H_6 = \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

$$H_7 = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{matrix} \qquad H_8 = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{matrix} \qquad H_9 = \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$$

$$H_{10} = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{matrix} \qquad H_{11} = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{matrix} \qquad H_{12} = \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$$

$$H_{13} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \qquad H_{14} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \qquad H_{15} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad (56)$$

where the square of any matrix in the above set equals ± times the identity matrix and the product of any two matrices in the set is proportional to a matrix in the set of 16. (These matrices H^{i}_{j} are obtained by considering an arbitrary antisymmetric or symmetric matrix, with zero diagonal elements, raising one of the indices using (11) and then applying the condition $H^{i}_{\alpha}H^{\alpha}_{j} = .25H^{\beta}_{\alpha}H^{\alpha}_{\beta}\delta^{i}_{j}$ .The following analysis applies regardless of the value given to c, so for simplicity c is chosen equal to 1. The matrices with nonzero diagonal elements are chosen so that $H^{\alpha}_{\alpha} = 0$) There are 6 independent matrices derived from the anti- symmetric matrix (H_i i =1,2,...6), 6 independent matrices derived from the symmetric matrix (H_i i = 7,8,9....12) and 3 independent matrices derived from the matrix with nonzero diagonal elements (H_i i = 13,14,15). As will be shown, these matrices endow the particles whose solutions are given by (55) with spin. It can also be shown that all of the matrices in column 1 of the above matrices are rotationally symmetric about the z axis, that all of the matrices in column 2 of the above matrices are rotationally symmetric about the y axis and that all the matrices in column 3 of the above matrices are rotationally symmetric about the x axis, where for instance a rotation through angle Θ about the z axis is given by

$$\begin{matrix} x^B = & \cos\Theta & \sin\Theta & 0 & 0 & x \\ y^B = & -\sin\Theta & \cos\Theta & 0 & 0 & y \\ z^B = & 0 & 0 & 1 & 0 & z \\ t^B = & 0 & 0 & 0 & 1 & t \end{matrix}$$

where B represents the transformed axes. Also matrices 4,5,6 are the complex conjugates of matrices 1,2,3 and matrices 10,11,12 are the complex conjugates of matrices 7,8,9. Matrices 13,14,15 are their own complex conjugates.

If (55) is inserted into (49), (50), (51) and the coefficients of like matrices are grouped, then the following differential equations are obtained for the scalar dependent variables f_N, N = 0,1,2,.....15

$$\Box^2 f_0 + \alpha^{*2} f_0 = \beta(-f_0^2 + f_1^2 + f_2^2 - f_3^2 + f_4^2 + f_5^2 - f_6^2 - f_7^2 - f_8^2 + f_9^2 - f^B{}_{10}{}^2 - f_{11}^2 + f_{12}^2 - f_{13}^2 - f_{14}^2 - f_{15}^2)$$
$$\Box^2 f_1 + \alpha^{*2} f_1 = -2\beta(f_0f_1 + f_4f_{13} - if_5f_{12} + if_6f_{11})$$
$$\Box^2 f_2 + \alpha^{*2} f_2 = -2\beta(f_0f_2 + if_4f_9 + f_5f_{14} - if_6f^B{}_{10})$$
$$\Box^2 f_3 + \alpha^{*2} f_3 = -2\beta(f_0f_3 + if_4f_8 - if_5f_7 + f_6f_{15})$$
$$\Box^2 f_4 + \alpha^{*2} f_4 = -2\beta(f_0f_4 + f_1f_{13} + if_2f_9 - if_3f_8)$$
$$\Box^2 f_5 + \alpha^{*2} f_5 = -2\beta(f_0f_5 - if_1f_{12} + f_2f_{14} + if_3f_7)$$
$$\Box^2 f_6 + \alpha^{*2} f_6 = -2\beta(f_0f_6 - if_1f_{11} + if_2f^B{}_{10} + f_3f_{15})$$
$$\Box^2 f_7 = -2\beta(f_0f_7 - if_3f_5 + if_9f_{11} + f^B{}_{10}f_{13})$$
$$\Box^2 f_8 = -2\beta(f_0f_8 + if_3f_4 - if^B{}_{10}f_{12} + f_{11}f_{14})$$
$$\Box^2 f_9 = -2\beta(f_0f_9 + if_2f_4 - if_7f_{11} + f_{12}f_{15})$$
$$\Box^2 f^B{}_{10} = -2\beta(f_0f^B{}_{10} + if_2f_6 + f_7f_{13} - if_8f_{12})$$
$$\Box^2 f_{11} = -2\beta(f_0f_{11} - if_1f_6 + if_7f_9 + f_8f_{14})$$
$$\Box^2 f_{12} = -2\beta(f_0f_{12} - if_1f_5 + f_9f_{15} + if_8f^B{}_{10})$$
$$\Box^2 f_{13} = -2\beta(f_0f_{13} - f_1f_4 + f_7f^B{}_{10} + f_{14}f_{15})$$
$$\Box^2 f_{14} = -2\beta(f_0f_{14} - f_2f_5 + f_8f_{11} + f_{13}f_{15})$$
$$\Box^2 f_{15} = -2\beta(f_0f_{15} + f_3f_6 - f_9f_{12} + f_{13}f_{14}) \quad (57)$$

where $\alpha^* = m_0c^2/h$ and $\beta = m_0c^2/h^2$

If eight of the scalar dependent variables in the above equations are assumed to have a time independent part (denoted by subscript 0) and eight of the scalar dependent variables in the above equations are assumed to lack a time independent part, (lack a subscript 0) there are exactly fifteen different sets of coupled partial differential equations with each equation in the set possessing the same form as either (2) or (3). If the variable on the left side of a scalar equation has a time independent part then the scalar equation form is similar to (2). Specifically, the quadratically nonlinear terms on the right side of the equation consist of products of two variables with time independent parts like V^2 or products of two variables that lack a time independent part like Ψ^2 . If the dependent variable on the left side of a scalar equation lacks a time independent part, the equation form is similar to (3). Specifically, the quadratically nonlinear terms on the right side of the equation consist of the products of two variables, one of which contains a time independent part and one of which lacks a time independent part, like $V\Psi$. For example

$$\begin{aligned}
\Box^2 f_0 + \alpha^{*2} f_0 &= \beta(-f_0^2 + f_{10}^2 + f_2^2 - f_3^2 + f_{40}^2 + f_{50}^2 - f_{60}^2 - f_7^2 - f_8^2 + f_9^2 - f^B{}_{10}^2 - f_{110}^2 + f_{120}^2 \\
&\quad - f_{130}^2 - f_{14}^2 - f_{15}^2) \\
\Box^2 f_{10} + \alpha^{*2} f_{10} &= -2\beta(f_0 f_{10} + f_{40} f_{130} - i f_{50} f_{120} + i f_{60} f_{110}) \\
\Box^2 f_2 + \alpha^{*2} f_2 &= -2\beta(f_0 f_2 + i f_{40} f_9 + f_{50} f_{14} - i f_{60} f^B{}_{10}) \\
\Box^2 f_3 + \alpha^{*2} f_3 &= -2\beta(f_0 f_3 + i f_{40} f_8 - i f_{50} f_7 + f_{60} f_{15}) \\
\Box^2 f_{40} + \alpha^{*2} f_{40} &= -2\beta(f_0 f_{40} + f_{10} f_{130} + i f_2 f_9 - i f_3 f_8) \\
\Box^2 f_{50} + \alpha^{*2} f_{50} &= -2\beta(f_0 f_{50} - i f_{10} f_{120} + f_2 f_{14} + i f_3 f_7) \\
\Box^2 f_{60} + \alpha^{*2} f_{60} &= -2\beta(f_0 f_{60} - i f_{10} f_{110} + i f_2 f^B{}_{10} + f_3 f_{15}) \\
\Box^2 f_7 &= -2\beta(f_0 f_7 - i f_3 f_{50} + i f_9 f_{110} + f^B{}_{10} f_{130}) \\
\Box^2 f_8 &= -2\beta(f_0 f_8 + i f_3 f_{40} - i f^B{}_{10} f_{120} + f_{110} f_{14}) \\
\Box^2 f_9 &= -2\beta(f_0 f_9 + i f_2 f_{40} - i f_7 f_{110} + f_{120} f_{15}) \\
\Box^2 f^B{}_{10} &= -2\beta(f_0 f^B{}_{10} + i f_2 f_{60} + f_7 f_{130} - i f_8 f_{120}) \\
\Box^2 f_{110} &= -2\beta(f_0 f_{110} - i f_{10} f_{60} + i f_7 f_9 + f_8 f_{14}) \\
\Box^2 f_{120} &= -2\beta(f_0 f_{120} - i f_{10} f_{50} + f_9 f_{15} + i f_8 f^B{}_{10}) \\
\Box^2 f_{130} &= -2\beta(f_0 f_{130} - f_{10} f_{40} + f_7 f^B{}_{10} + f_{14} f_{15}) \\
\Box^2 f_{14} &= -2\beta(f_0 f_{14} - f_2 f_{50} + f_8 f_{110} + f_{130} f_{15}) \\
\Box^2 f_{15} &= -2\beta(f_0 f_{15} + f_3 f_{60} - f_9 f_{120} + f_{130} f_{14}) \qquad (58)
\end{aligned}$$

where the subscript zero indicates a variable with a time independent part, except for $f^B{}_{10}$ which lacks a time independent part. (If $f^B{}_{10}$ possessed a time independent part, it would have a double zero subscript like $f^B{}_{100}$)

(In the theory developed here, a particle is represented as a bound state of a field of potential energy, produced by the particle itself. In the rest frame shared by the potential energy field and the particle, if particle energy is quantized, then so is particle rest mass.)

See the following table which shows the 15 different solutions of (57) which possess the form described above, (case 3 in the table corresponds to the above case in (58)).

	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
1	0			0	0	0	0	0	0							0
2	0		0		0	0	0			0	0				0	
3	0	0			0	0	0					0	0	0		
4	0	0	0	0			0				0	0				0
5	0	0	0	0		0		0					0		0	
6	0	0	0	0	0				0	0				0		
7	0			0	0				0		0	0	0		0	
8	0		0		0			0		0		0	0			0
9	0			0		0		0		0	0	0		0		
10	0	0					0	0	0	0		0			0	
11	0	0				0			0	0	0		0			0
12	0		0				0	0	0		0		0	0		
13	0		0			0			0			0		0	0	0
14	0			0			0			0			0	0	0	0
15	0	0			0			0			0			0	0	0
16	0	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

Table 1

where 0 indicates a dependent variable with a time independent part and a space with no entry indicates that the dependent variable lacks a time independent part. Case 16 in the above table corresponds to the scalar solution of (17) already discussed, and represents a structureless spin 0 boson. The x's in case 16 indicate that there is no variable at all in that case. The cases are grouped into five sets, with the first set containing three cases, the second set containing three cases, the third set containing six cases, the fourth set containing three cases and the fifth set containing one case.

It can be shown that a member of one set above can be transformed into another member of the same set by a ninety degree rotation about one of the coordinate axes and that the cases in two different sets can't be transformed into each other by rotation. Therefore the particle types in each set are physically distinct This result is accomplished using (55) and the transformation equations of the mixed contravariant, covariant tensors (56), which are

$$H^{B\,i}{}_{j} = (\partial x^{B\,i}/\partial x^{\alpha})(\partial x^{\beta}/\partial x^{B\,j})H^{\alpha}{}_{\beta} \qquad (59)$$

where x^i , i = 1,2,3,4 are the untransformed axes and $x^{B\,j}$, j=1,2,3,4 are the transformed axes. For instance, consider a 90 degree rotation about the x axis, given by $x^B = x$, $y^B = z$, $z^B = -y$, $t^B = t$, where x,y,z,t represent the initial axes and x^B, y^B, z^B, t^B represent the rotated axes. Applying this transformation and (59) to matrix H_1, transforms H_1 into $-H_2$. When this is done to all the contravariant covariant matrices in the expression for case 1, keeping track of whether the scalars have a time independent part or not, case 1 is rotated into an expression with the same form as case 2. The same thing can be done with rotations about the y and z axes and for all the cases in all the groups in the chart.

It is possible to identify matrices H_1 , H_2 , H_3 with the components of a spin vector and matrices H_4 , H_5 , H_6 with the components of another spin vector where

$$\sigma_x = ½\, H_3 ,\ \sigma_y = ½\, iH_2, \sigma_z = ½\, iH_1 \qquad (60)$$

$$\sigma^B{}_x = -½\, H_6 ,\ \sigma^B{}_y = -½\, iH_5 ,\ \sigma^B{}_z = -½\, iH_4 \qquad (61)$$

and these obey the commutation relations

$$[\sigma_x, \sigma_y] = i\sigma_z ,\ [\sigma_x,\sigma_z] = -i\sigma_y ,\ [\sigma_y, \sigma_z] = i\sigma_z \qquad (62)$$

$$[\sigma^B{}_x, \sigma^B{}_y] = i\sigma^B{}_z ,\ [\sigma^B{}_x , \sigma^B{}_z] = -i\sigma^B{}_y ,\ [\sigma^B{}_y, \sigma^B{}_z] = i\sigma^B{}_x \qquad (63)$$

The other matrices in (56) satisfy anticommutation relations and can't be associated with particle spin.

Matrices H_i , i=1,2,...6 are therefore associated with particle spin. The functions f_{i0}, i = 1,2,...,6 that possess a time independent part are gravitational self potential energy functions associated with particle spin. When these functions are inserted into (48) they should determine particle electroweak properties. The functions f_i , i = 1,2,....,6 that lack a time independent part are the wave functions associated with particle spin. To a first order of approximation, if the quadratically nonlinear terms on the right side of (57) and (58) are ignored, then f_{i0} , i = 1,2,...,6 have a time independent part that satisfies $\nabla^2 f_{i0} - \alpha^2 f_{i0} = 0$, $\alpha = m_0 c/h$ and f_i , i = 1,2,...,6 that lack a time independent part satisfy $\Box^2 f_i + \alpha^{*2} f_i = 0$, where α^* was previously defined. In order to later show the rotational symmetry of the spin part of the solutions in table 1, it is assumed that

$$f_{i0} = g_1(R) \ \text{and}\ f_i = g_2(R,t) ,\ i = 1,2,....6 \qquad (64)$$

The first derivatives of f_{i0} and f_i with respect to coordinates are also solutions of the differential equations satisfied by the f_i and f_{i0} where

$\partial f_{i0}/\partial x = (\partial g_1/\partial R)(\partial R/\partial x) = (\partial g_1/\partial R)(x/R)$, $\partial f_{i0}/\partial y = \ldots\ldots$, etc and

$$\partial f_i/\partial x = (\partial g_2/\partial R)(\partial R/\partial x) = (\partial g_2/\partial R)(x/R)\,,\ \partial f_i/\partial y = \ldots\ldots, \qquad i=1,2,\ldots..6 \quad (65)$$

For instance in case 3, the spin related, time independent part of U^i_j is given by

$$U_0{}^i_j = f_{10}H_1 + f_{40}H_4 + f_{50}H_5 + f_{60}H_6$$
$$=(1/R)(\partial g_1/\partial R)(-izH_1 - izH_4 + iyH_5 + xH_6) \qquad (66)$$

The coefficients of the matrices H_1 , H_4 , H_5 , H_6 have been chosen to be consistent with (65) and to preserve rotational symmetry about the z axis, and it can be shown that U_0 is rotationally symmetric about the z axis. In table 1, case 3 represents the only state in the first set where U_0 is rotationally symmetric about the z axis and it is the only state for which the z component of its spin vector is in an eigenstate. The time independent part of U for cases 1 and 2 in table 1 are rotationally symmetric about the x and y axes respectively and it is assumed that these states are not observable. However, it is possible to form states with time independent parts which are rotationally symmetric about the z axis by adding states 1 and 2 and by adding states 1, 2 and 3.

The time dependent, spin related part of the wave function in case 3 is given by

$$\Psi = (1/R)(\partial g_2/\partial R)(-iyH_2 + xH_3) \qquad (67)$$

Forming the determinant of this matrix it can be shown that its eigenvalues are

$$\Lambda_{1,2} = \pm[(1/R)(\partial g_2/\partial R)]^{.5}(x^2 + y^2)^{.5} \qquad (68)$$

and it is inferred that case 3 represents a spin ½ particle. This suggests that the particles in states 1,2,3 represent spin ½ leptons. The particles in states 4,5,6 are obtained from states 1,2,3 by complex conjugation, and it is inferred that states 4,5,6 are the antiparticles of states 1,2,3.

Similarly it can be shown that the time independent, spin dependent part of the solution in case 15 is given by

$$U_0 = (1/R)(\partial g_1/\partial R)(zH_1 + zH_4) \qquad (69)$$

this potential energy function is clearly rotationally symmetric about the z axis since as previously stated H_1 and H_4 are rotationally symmetric about the z axis. Also of the cases 13, 14, 15 only case 15 has spin dependent, time independent part which is rotationally symmetric about the z axis. The time independent wave function for cases 13 and 14 are rotationally symmetric about the y and x axes respectively. Therefore of the three cases, only case 15 has a z component of the particle's spin vector which is in an eigenstate. States for which the time independent wave function are rotationally symmetric about the z axis can be formed by adding states 13 and 14 and by adding states 13, 14 and 15.

The time dependent part of the wave function in case 15 is given by

$$\Psi = (1/R)(\partial g_2/\partial R)(-iyH_2 + xH_3 + iyH_5 + xH_6) \quad (70)$$

The eigenvalues associated with this matrix are

$$\Lambda_{1,2,3} = 0,\ \pm[(1/R)(\partial g_2/\partial R)]^{.5}\,(x^2 + y^2)^{.5} \quad (71)$$

And it is inferred that case 15 represents a spin 1 particle. This suggests that the particles represented by states 13,14,15 in table 1 represent spin 1 bosons.

It can be shown that none of the spin dependent, time independent potential energy functions for cases 7 to 12 are rotationally symmetric about the z axis. Therefore, the z component of the spin in none of cases 7 to 12 is in an eigenstate, and none of these states can represent an independent observable particle. For instamce in case 7, the spin dependent, part of the potential energy function is given by

$$U_0 = G_1(xH_3 + izH_4) \quad (72)$$

and a 90 degree rotation about the z axis carries this expression into barred coordinates

$$U_0 = G_1(i\,\bar{y}H_2 + i\,\bar{z}H_4) \quad (73)$$

which is not the same as the original function. It is possible however, to form particle states which are the sum of three particle states from cases 7 to 12 or the sum of a particle state and its complex conjugate (or antiparticle state) from cases 7 to 12, whose spin dependent, time independent potential energy functions are rotationally summetric about the z axis. Therefore these composite states can represent a physically independent and observable particle.

For instance the sum of the potential energy functions for cases 8, 9, 10 and for cases 7, 11, 12 is

$$U_0 = G_1(-izH_1 - iyH_2 + xH_3 + izH_4 + iyH_5 + xH_6) \quad (74)$$

and it can be shown that this function is rotationally symmetric about the z axis. Therefore the z component of the spin vector of this particle would be in an eigenstate and the associated wave function could represent a physically observable particle.
The wave function for this composite is given by

$$\Psi = 2G_2(R)(-izH_1 - iyH_2 + xH_3 + izH_4 + iyH_5 + xH_6) \quad (75)$$

This is a particle whose spin related wave function is equal to the complex conjugate of its spin related wave function. To preclude this possibility, the coefficients of H_4, H_5, H_6 are set to zero, and it can be shown that this wave function represents a spin ½ particle. If the coefficients of H_4, H_5, H_6 are reduced by ½ instead, it can be shown that the associated wave function represents a spin 3/2 particle.

The potential energy function for the sum of cases 9 and its complex conjugate or antiparticle and the sum of cases 12 and its complex conjugate or antiparticle , can be shown to be rotationally symmetric about the z axis. The spin related wave function associated with these combination states can be shown to represent spin 1 particles. Also it can be shown that the spin related wave function associated with the sum of case 9 and the negative of the complex conjugate of case 12 (where case 12 represents the complex conjugate of state 9) can be made to vanish. Therefore this combination state represents a spin 0 particle.

These arguments suggest that the particles in states 7 to 12 in table 1 represent quarks.

Strength and Range of Elementary Particle Gravitational Self Interaction

The previous analysis shows how to distinguish between spin ½ leptons, spin ½ quarks and spin 0 and spin 1 bosons. It does not show why the interaction strength between leptons and bosons is so much weaker than the interaction strength between quarks. In this section it is argued that the reason for the difference in interaction strength between leptons and quarks is that the gravitational self interaction strength at short range of quarks is much stronger than the gravitational self interaction strength at short range of leptons.

It was shown in the previous section that the scalar functions f_i and f_{i0} $i = 1,2,\ldots,6$ are spin related. The f_{i0} $i = 1,2,\ldots 6$ which possess a time independent part, must be chosen to maintain rotational symmetry about the z axis so that the z component of particle spin can be in an eigenstate. When the nonlinear terms on the right sides of the equations they satisfy, in (57) and (58), are ignored in all but the first equation of each set, then to the first order of approximation, the f_{i0} satisfy

$$\nabla^2 f_{i0} - \alpha^2 f_{i0} = 0, \alpha = m_0 c/h \text{ , } i= 1,2,\ldots 6 \qquad (76)$$

Equation (76) has a solution of the following form

$$f_{i0} = A_i e^{-\alpha R}/R, \ \ i = 1,2,\ldots.6 \qquad (77)$$

where the A_i are constants. The derivatives of f_0 with respect to coordinates x,y,z are also solutions of (76). For instance

$$\partial f_{i0} / \partial x = -Ae^{-\alpha R}(1/R^2 + \alpha/R) \bullet (x/R)$$

is a solution of (76). It was shown in the previous section that if the time independent spin related part of the gravitational self potential energy is to be rotationally symmetric about the z axis that the derivative solutions of (76) must be used. It is assumed that these f_{i0} $i = 1,2,\ldots.6$ satisfy (48) and determine particle time independent electroweak properties.

The nonspin related, time independent scalar functions f_{i0} , $i = 7,8, \ldots, 15$ in all cases in Table 1 satisfy

$$\nabla^2 f_{i0} = 0 \ , \ i = 7,8,\ldots,15 \qquad (78)$$

when the right sides of the equations they satisfy in (57) and (58) are ignored to first order of approximation. It is assumed that the functions f_{i0} , $i = 7,8, \ ,15$ can be chosen so that the gravitational self potential energy represented by f_0 of all of the particles in table 1, possess a nonspin related part which is spherically symmetric or isotropic about the origin. The f_{i0} have the solutions

$$f_{i0} = A_i/R, \ i = 7,8,\ldots,15 \qquad (79)$$

where the A_i are constants. The fields these solutions represent, like the time independent electrostatic and time independent gravitational field of a point particle, decline as 1/R where R is the distance from the particle. No such fields which decline as 1/R other than the electrostatic and time independent gravitational field have ever been observed. These solutions for f_{i0} $i = 7,8,\ldots,15$ are therefore unsatisfactory.

The derivatives of f_{i0} with respect to the coordinates x,y,z are also solutions of $\nabla^2 f_{i0} = 0$ and these solutions decline as $1/R^2$ with respect to the distance R from the particle. For instance

$$f_{i0} = (\partial/\partial x)(A_i/R) = -A_i x/R^3 \qquad (80)$$

These solutions represent the gravitational self interaction potential energy of the particle and also its interaction potential energy with other particles. The equation for the nonspin related time independent gravitational self potential energy for case 3 in table 1 is given by

$$U_0{}^i{}_j = f_{110} H_{11}{}^i{}_j + f_{120} H_{12}{}^i{}_j + f_{130} H_{13}{}^i{}_j$$

where case 3 in table 1, as shown in the previous section, represents a lepton whose spin related potential energy is rotationally symmetric about the z axis and therefore represents a possible lepton state.

It is further assumed that

$$f_{110} = -iAy/R^3 , \ f_{120} = -Ax/R^3 , \ f_{130} = -iAz/R^3 \qquad (81)$$

where $i = \sqrt{-1}$. When these are inserted into the time independent version of the first of (58), ignoring all other f_{i0} and f_i and ignoring $f_0{}^2$ on the right side of this equation, the differential equation for f_0 has a right side which is a function R alone and proportional to $1/R^4$. Solving this differential equation for f_0 gives

$$f_0 \text{ proportional to } 1/R^2 \quad (82)$$

at short range where f_0 represents the isotropic nonspin related part of the particle's gravitational self interaction potential energy.

It was shown in the previous section that a composite state composed of states 8,9,10 or 7, 11,12 in table 1 might represent a physical particle, a hadron. The nonspin related part of the gravitational self potential energy function which is the sum of cases 8,9,10 from table 1 is given by

$$U_0 = (f^8{}_{70} + f^9{}_{70} + f^{10}{}_{70})H_7 + f^{10}{}_{80} H_8 + (f^8{}_{90} + f^9{}_{90} + f^{10}{}_{90})H_9 + f^9{}_{100} H_{10}$$

$$+ (f^8{}_{110} + f^9{}_{110} + f^{10}{}_{110})H_{11} + f^8{}_{120} H_{12} + f^9{}_{130} H_{13} + f^{10}{}_{140} H_{14} + f^8{}_{150} H_{15} \quad (83)$$

where the superscript references the particle state in table 1.

Equation (83) can be rewritten in the more compact form as

$$U_0 = f_{70} H_7 + f_{80} H_8 + f_{90} H_9 + f_{100} H_{10} + f_{110} H_{11} + f_{120} H_{12} + f_{130} H_{13} + f_{140} H_{14} + f_{150} H_{15} \quad (84)$$

Just as it was possible to form three new f_{i0} by differentiating $f_{i0} = A_i/R = \varphi$ in (79) with respect to the coordinates x,y,z, giving the f_{i0} in (81), it is possible to form new f_{i0} by differentiating each f_{i0} in (81) with respect to the coordinates x,y,z and associating these functions with the f_{i0} in (84) as follows:

$$f_{70} = \partial^2\varphi/\partial x^2 = A(-1/R^3 + 3x^2/R^5)$$
$$f_{80} = \partial^2\varphi/\partial x\partial y = A(3xy/R^5)$$
$$f_{90} = \partial^2\varphi/\partial x\partial z = Ai(3xz/R^5)$$
$$f_{100} = \partial^2\varphi/\partial y\partial x = A(3xy/R^5)$$
$$f_{110} = \partial^2\varphi/\partial y^2 = A(-1/R^3 + 3y^2/R^5)$$
$$f_{120} = \partial^2\varphi/\partial z\partial y = Ai(3yz/R^5)$$
$$f_{130} = \partial^2\varphi/\partial x\partial z = A(3xz/R^5)$$
$$f_{140} = \partial^2\varphi/\partial y\partial z = A(3yz/R^5)$$
$$f_{150} = \partial^2\varphi/\partial z^2 = A(-1/R^3 + 3z^2/R^5) \quad (85)$$

Inserting (85) into the time independent version of the first of equation (57) gives

$$\nabla^2 f_0 - \alpha^2 f_0 = (m_0/h^2)(f_{70}^2 + f_{80}^2 - f_{90}^2 + f_{100}^2 + f_{110}^2 - f_{120}^2 + f_{130}^2 + f_{140}^2 + f_{150}^2)$$

$$= 6m_0A^2/(h^2R^6) \quad (86)$$

The solution for f_0 at short range is

$$f_0 \text{ proportional to } 1/R^4$$

The self interaction potential energy of leptons at short range is proportional to

$1/R^2$ and the self interaction potential energy of quarks at short range is proportional to $1/R^4$. It is inferred that the interaction strength between leptons at short range is proportional to $1/R^2$ and the interaction strength between quarks at short range is proportional to $1/R^4$.

Conclusions

The interactions of any system of elementary particles can be represented by the coupled covariant partial differential equations of a single gravitationally self interacting particle, evolving in a curved background independent space consisting of three spatial and one temporal dimension. The covariant equations couple quantum mechanics, general relativity and the equations of the electroweak and strong fields. The field of gravitational self potential energy endows the particles in the system with rest mass, and affects their macroscopic motion. The rotational stability of the galaxy is explained without the need to assume the existence of dark matter. The theory developed here may facilitate the study of molecular interactions which are the basis of drug research, energy research and materials research.

Appendix 1. Stress energy tensor of the electromagnetic field in polar coordinates in flat spacetime:

$$\Phi_{1\,j}^{\;i} = \begin{array}{cccc}
c^4(E_1^2-E_2^2-E_3^2+H_1^2-H_2^2-H_3^2)/2 & c^4R(E_1E_2+H_1H_2) & c^4R\sin(\theta)(E_1E_3+H_1H_3) & c^5(E_2H_3-H_2E_3) \\
c^4(E_1E_2+H_1H_2)/R & c^4(-E_1^2+E_2-E_3^2-H_1^2+H_2^2-H_3^2)/2 & c^4\sin(\theta)(E_2E_3+H_2H_3) & c^5(H_1E_3-E_1H_3)/R \\
c^4(E_1E_3+H_1H_3)/(R\sin(\theta)) & c^4(E_2E_3+H_2H_3)/\sin(\theta) & c^4(-E_1^2-E_2^2+E_3^2-H_1^2-H_2^2+H_3^2)/2 & c^5(E_1H_2-H_1E_2)/(R\sin(\theta)) \\
c^3(H_2E_3-E_2H_3) & c^3R(E_1H_3-H_1E_3) & c^3R\sin(\theta)(H_1E_2-E_1H_2) & c^4(E_1^2+E_2^2+E_3^2+H_1^2+H_2^2H_3^2)/2
\end{array}$$

Where $E_1 = E_R$, $E_2 = E_\theta$, $E_3 = E_\varphi$, $H_1 = H_R$, $H_2 = H_\theta$, $H_3 = H_\varphi$ (Physical Components)

Appendix 2 Equations (61) with electromagnetic substitution in polar coordinates in flat spacetime

$$c^4\partial/\partial R(E_1^2-E_2^2-E_3^2+H_1^2-H_2^2-H_3^2)/2+(c^4/R)\partial/\partial\theta(E_1E_2+H_1H_2)+ (c_4/(R\sin(\theta)))\partial/\partial\varphi(E_1E_3+H_1H_3)+c^4(2E_1^2-E_2^2-E_3^2+2H_1^2-H_2^2-H_3^2)/R + c^4\mathrm{ctn}(\theta)(E_1E_2+H_1H_2)/R+c^3\partial/\partial t(H_2E_3-E_2H_3) = (m_0c^8/(2kh^2))(\partial/\partial R)f_{01} \qquad \text{B.1}$$

$$c^4\partial/\partial R(R(E_1E_2+H_1H_2))-c^4\partial/\partial\theta(E_1^2-E_2^2+E_3^2+H_1^2-H_2^2+H_3^2)/2+ (c^4/\sin(\theta))\partial/\partial\varphi(E_2E_3+H_2H_3)+2c^4(E_1E_2+H_1H_2)+c^4\mathrm{ctn}(\theta)(E_2^2-E_3^2+H_2^2-H_3^2) + c^3R\partial/\partial t(E_1H_3-H_1E_3) = (m_0c^8/(2kh^2))(\partial/\partial\theta)f_{01} \qquad \text{B.2}$$

$$c^4\sin(\theta)\partial/\partial R(R(E_1E_3+H_1H_3))+c^4\partial/\partial\theta(\sin(\theta)(E_2E_3+H_2H_3)) - c^4\partial/\partial\varphi(E_1^2+E_2^2-E_3^2+H_1^2+H_2^2-H_3^2)+c^4\cos(\theta)(E_2E_3+H_2H_3) + 2c^4\sin(\theta)(E_1E_3+H_1H_3)+c^3R\sin(\theta)\partial/\partial t(H_1E_2-E_1H_2) = (m_0c^8/(2kh^2))(\partial/\partial\varphi)f_{01} \qquad \text{B.3}$$

$$c^5\partial/\partial R(H_3E_2-H_2E_3) + (c^5/R)\partial/\partial\theta(H_1E_3- E_1H_3) + (c^5/(R\sin(\theta)))\partial/\partial\varphi(E_1H_2-H_1E_2) + c^4\partial/\partial t(E_1^2+E_2^2+E_3^2+H_1^2+H_2^2+H_3^2)/2+2c^5(H_3E_2-H_2E_3)/R + (c^5\mathrm{ctn}(\theta)/R)(H_1E_3-E_1H_3) = (m_0c^8/(2kh^2))(\partial/\partial t)f_{01} \qquad \text{B.4}$$

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