Proof of convergence of the Fourier series

Pawel Jan Piskorz

Abstract

Absolute and uniform convergence of any Fourier series has been proven using integration with substitution of variables and limits and the Dirichlet integral value. Our proof of the convergence of the Fourier series requires direct computations.

I Introduction

Let us consider partial sum $s_N(x)$ of the Fourier series

$$s_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N}(a_k \cos kx + b_k \sin kx) \tag{1}$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \tag{2}$$

for $k = 0, 1, 2, \ldots, N$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx \tag{3}$$

for $k = 1, 2, \ldots, N$.

Substituting the expressions for the Fourier coefficients we obtain

$$\begin{aligned} s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt + \frac{1}{\pi}\sum_{k=1}^{N}\Big[\int_{-\pi}^{\pi} f(t)\cos kt\, dt \cdot \cos kx + \\ + \int_{-\pi}^{\pi} f(t)\sin kt\, dt \cdot \sin kx\Big] \\ = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{N}(\cos kt \cos kx + \sin kt \sin kx)\Big]\, dt = \\ = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{N}(\cos k(t-x))\Big]\, dt \end{aligned} \tag{4}$$

II Sum of cosines formula derivation

We use a formula for the sum of the geometric series $\sum_{k=1}^{N} e^{ik\varphi}$ as in [1] and for convenience presented with modifications in Section V.

$$\sum_{k=1}^{N} e^{ik\varphi} = \sum_{k=1}^{N} (\cos k\varphi + i \sin k\varphi) \tag{5}$$

$$= \cos((N+1)\varphi/2) \frac{\sin N\varphi/2}{\sin \varphi/2} + i \sin((N+1)\varphi/2) \frac{\sin N\varphi/2}{\sin \varphi/2}$$

where $i^2 = -1$. Computing from equations

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \tag{6}$$

and

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta \tag{7}$$

we obtain

$$\cos\alpha \sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta)) \tag{8}$$

$$\sin\alpha \sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta)) \tag{9}$$

and we can substitute

$$\alpha = (N+1)\varphi/2 \qquad \beta = N\varphi/2 \tag{10}$$

$$\alpha + \beta = N\varphi + \varphi/2 \qquad \alpha - \beta = \varphi/2 \tag{11}$$

After computations we arrive at

$$\sum_{k=1}^{N} \sin k\varphi = -\frac{1}{2}(\cos(N\varphi + \varphi/2) - \cos\varphi/2) \frac{1}{\sin\varphi/2} \tag{12}$$

$$\sum_{k=1}^{N} \cos k\varphi = \frac{1}{2}(\sin(N\varphi + \varphi/2) - \sin\varphi/2) \frac{1}{\sin\varphi/2} \tag{13}$$

From equation (13) we receive

$$\frac{1}{2} + \sum_{k=1}^{N} \cos k\varphi = \frac{\sin(N\varphi + \varphi/2)}{2\sin\varphi/2} \tag{14}$$

III Applying the sum of cosines formula

We apply the formula from equation (14) for the sum of cosines having $\varphi = t - x$ in equation (4)

$$s_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin(N(t-x)+(t-x)/2)}{2\sin(t-x)/2}\,dt \tag{15}$$

We substitute $t - x = u$. Then we have the result obtained in [2]

$$\begin{aligned} s_N(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\,\frac{\sin(Nu+u/2)}{2\sin u/2}\,du = \\ = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(Nu+u/2)}{2\sin u/2}\,du \end{aligned} \tag{16}$$

Now we substitute in equation (16) $Nu = \alpha$. Then we have $u = \alpha/N$, $u/2 = \alpha/2N$, $du = d\alpha/N$. For the limits of integration if $u = -\pi$ then $\alpha = -N\pi$ and if $u = \pi$ then $\alpha = N\pi$. In this way we obtain

$$\begin{aligned} s_N(x) = \frac{1}{\pi}\int_{-N\pi}^{N\pi} f(x+\alpha/N)\frac{\sin(\alpha+\alpha/2N)}{2N\sin\alpha/2N}\,d\alpha = \\ = \frac{1}{\pi}\int_{-N\pi}^{N\pi} f(x+\alpha/N)\frac{\sin(\alpha+\alpha/2N)}{\alpha(\sin\alpha/2N)/(\alpha/2N)}\,d\alpha \end{aligned} \tag{17}$$

IV Taking the limit and the final result

In particular for $N \to \infty$

$$\begin{aligned} \lim_{N\to\infty} s_N(x) = \\ = \lim_{N\to\infty}\frac{1}{\pi}\int_{-N\pi}^{N\pi} f(x+\alpha/N)\frac{\sin(\alpha+\alpha/2N)}{\alpha(\sin\alpha/2N)/(\alpha/2N)}\,d\alpha \end{aligned} \tag{18}$$

We notice that in the numerator above

$$\lim_{N\to\infty}\sin(\alpha+\alpha/2N) = \sin\alpha \tag{19}$$

and in the denominator we have the limit

$$\lim_{N\to\infty}\alpha(\sin\alpha/2N)/(\alpha/2N) = \alpha \tag{20}$$

what is in agreement with the known limit

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{21}$$

where $\theta = \alpha/2N$.

In this way we receive

$$s(x) = \lim_{N\to\infty} s_N(x) = \tag{22}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{\sin\alpha}{\alpha}\,d\alpha = \frac{1}{\pi}f(x)\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha =$$
$$= \frac{1}{\pi}f(x)\pi = f(x)$$

We have shown that the partial sum $s_N(x)$ of any Fourier series for $N \to \infty$ converges uniformly to the function $f(x)$ for all values of x except in the points of $f(x)$ discontinuity. The series $s(x)$ is uniformly convergent in the range $[a, b]$ because for any $\epsilon > 0$ one can select such index M independent on the value of x that for any $N \geq M$ it occurs

$$|s_N(x) - s(x)| < \epsilon \tag{23}$$

We obviously also have that

$$|s(x)| = \lim_{N\to\infty} |s_N(x)| = |f(x)| \tag{24}$$

what means that for any $\epsilon > 0$ one can select such index M independent on the value of x that for any $N \geq M$ it occurs

$$||s_N(x)| - |s(x)|| < \epsilon \tag{25}$$

The inequality above is the statement of absolute convergence of the Fourier series $s(x)$.

The conclusion is that if a series is a Fourier series of any periodic function $f(x)$ then it is convergent uniformly and absolutely to the function $f(x)$ except in the points of $f(x)$ discontinuity.

To compute the Dirichlet integral

$$\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha$$

we have used the result from Sections V and VI.

V Geometric series evaluation

Here we evaluate the sum of the geometric series as in [1]

$$S_N = \sum_{k=1}^{N} e^{ik\varphi} \tag{26}$$

We write

$$S_N = e^{i\varphi} + e^{i2\varphi} + e^{i3\varphi} + \ldots + e^{iN\varphi} \tag{27}$$

We multiply the above equation by $e^{i\varphi}$ on both sides obtaining

$$e^{i\varphi} S_N = e^{i2\varphi} + e^{i3\varphi} + \ldots + e^{iN\varphi} + e^{i(N+1)\varphi} \tag{28}$$

Now we subtract equation (28) from equation (27) and we receive

$$S_N - e^{i\varphi} S_N = e^{i\varphi} - e^{i(N+1)\varphi} \tag{29}$$

what can be rewritten as

$$S_N(1 - e^{i\varphi}) = e^{i\varphi} - e^{i(N+1)\varphi} \tag{30}$$

From equation (30) we compute the partial sum S_N as

$$\begin{aligned} S_N = \frac{e^{i\varphi} - e^{i(N+1)\varphi}}{1 - e^{i\varphi}} = \frac{e^{i(N+1)\varphi} - e^{i\varphi}}{e^{i\varphi} - 1} = e^{i\varphi} \frac{e^{iN\varphi} - 1}{e^{i\varphi} - 1} = \\ = e^{i\varphi} \frac{e^{iN\varphi} - 1}{e^{i\varphi/2}\left(e^{i\varphi/2} - e^{-i\varphi/2}\right)} = e^{i\varphi/2} \frac{e^{iN\varphi} - 1}{e^{i\varphi/2} - e^{-i\varphi/2}} = \\ = e^{i\varphi/2} e^{iN\varphi/2} \frac{e^{iN\varphi/2} - e^{-iN\varphi/2}}{e^{i\varphi/2} - e^{-i\varphi/2}} = e^{i(N\varphi+\varphi)/2} \frac{e^{iN\varphi/2} - e^{-iN\varphi/2}}{e^{i\varphi/2} - e^{-i\varphi/2}} = \\ = e^{i(N\varphi+\varphi)/2} \frac{\sin(N\varphi/2)}{\sin(\varphi/2)} \end{aligned} \tag{31}$$

Then we can write

$$S_N = \sum_{k=1}^{N} e^{ik\varphi} = e^{i(N\varphi+\varphi)/2} \frac{\sin(N\varphi/2)}{\sin(\varphi/2)} \tag{32}$$

and therefrom we obtain

$$\begin{aligned} S_N = \sum_{k=1}^{N} e^{ik\varphi} = \sum_{k=1}^{N} (\cos k\varphi + i \sin k\varphi) \\ = e^{i(N\varphi+\varphi)/2} \frac{\sin(N\varphi/2)}{\sin(\varphi/2)} = \\ = \cos((N+1)\varphi/2) \frac{\sin N\varphi/2}{\sin \varphi/2} + i \sin((N+1)\varphi/2) \frac{\sin N\varphi/2}{\sin \varphi/2} \end{aligned} \tag{33}$$

what gives

$$\sum_{k=1}^{N} \cos k\varphi = \cos((N+1)\varphi/2) \frac{\sin N\varphi/2}{\sin \varphi/2} \tag{34}$$

$$\sum_{k=1}^{N} \sin k\varphi = \sin((N+1)\varphi/2) \frac{\sin N\varphi/2}{\sin \varphi/2} \tag{35}$$

VI Dirichlet integral evaluation

We have written previously in Section II that

$$\frac{\sin(N\varphi+\varphi/2)}{2\sin\varphi/2}=\frac{1}{2}+\sum_{k=1}^{N}\cos k\varphi \tag{36}$$

We integrate both sides of the above equation from $-\pi$ to π

$$\int_{-\pi}^{\pi}\frac{\sin(N\varphi+\varphi/2)}{2\sin\varphi/2}\,d\varphi=\frac{1}{2}\int_{-\pi}^{\pi}d\varphi+\sum_{k=1}^{N}\int_{-\pi}^{\pi}\cos k\varphi\,d\varphi \tag{37}$$

receiving

$$\int_{-\pi}^{\pi}\frac{\sin(N\varphi+\varphi/2)}{2\sin\varphi/2}\,d\varphi=\pi \tag{38}$$

Now we substitute as before $N\varphi=\alpha$. Then we have $\varphi=\alpha/N$, $\varphi/2=\alpha/2N$, $d\varphi=d\alpha/N$. If $\varphi=-\pi$ then $\alpha=-N\pi$ and if $\varphi=\pi$ then $\alpha=N\pi$. After substitutions we may write that

$$\int_{-N\pi}^{N\pi}\frac{\sin(\alpha+\alpha/2N)}{2N\sin\alpha/2N}\,d\alpha=\pi \tag{39}$$

and after taking the limit N approaching infinity we arrive at

$$\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha=\pi \tag{40}$$

in a similar way like at the evaluation of the limit of partial sum of the Fourier series.

References

[1] Hirst Keith E (1995) *Modular Mathematics Numbers, Sequences and Series* Oxford: Butterworth-Heinemann

[2] Tolstov Georgi P Translated from the Russian by Silverman Richard A (1976) *Fourier Series* New York, NY: Dover Publications, Inc.