Hydrodynamics as U(1) gauge theory

Miftachul Hadi[1, 2]
[1)] *Badan Riset dan Inovasi Nasional (BRIN), KST Habibie (Puspiptek) Gd 442, Serpong, Tangerang Selatan 15314, Banten, Indonesia.*
[2)] *Institute of Mathematical Sciences, Kb Kopi, Jalan Nuri I, No.68, Pengasinan, Gn Sindur 16340, Bogor, Indonesia. E-mail: instmathsci.id@gmail.com*

We propose that hydrodynamics could be treated as $U(1)$ gauge theory where the velocity field written using Clebsch variables and the related vorticity are identical to the gauge potential and the field strength tensor, respectively.

Keywords: *hydrodynamics, $U(1)$ gauge theory, velocity field, vorticity, Clebsch variables.*

Let us observe[1]

$$C(\vec{v}) \equiv \int \varepsilon^{ijk} \ v^i \ \partial_j v^k \ dr = \int \vec{v} \cdot \vec{\omega} \ dr \tag{1}$$

where $C(\vec{v})$ is the fluid helicity, $\vec{v}$ is velocity field, $\vec{\omega}$ is vorticity.

In 3-dimensional space, the Chern-Simons integral could be written as[2–4]

$$h = \int_M \varepsilon^{\alpha\mu\nu} \ \vec{A}_\alpha \ \vec{F}_{\mu\nu} \ d^3r \tag{2}$$

where h is the electromagnetic helicity, a non-zero integer number (if h is zero it implies zero energy), M denotes 3-dimensional manifold, $\varepsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol, $\alpha, \mu, \nu = 1, 2, 3$ denote the 3-dimensional space, $\vec{A}_\alpha$ is the gauge potential, $\vec{F}_{\mu\nu}$ is the gauge field tensor[3] (the field strength tensor).

We consider that eqs. (1) and (2) are identical where velocity field and vorticity are identical to the gauge potential and the field strength tensor, respectively. It has a consequence that we could treat the hydrodynamics as the gauge theory.

In the simple case, let us assume that the hydrodynamical system is the fluid flow that has zero viscosity (an inviscid fluid) with non-zero vorticity in the Eulerian reference frame where the Clebsch representation for the velocity field is used. Then such a hydrodynamical system could be treated as an Abelian $U(1)$ gauge theory. This is because the structure of the velocity field in such a hydrodynamical system does not involve non-commuting interactions (there are no interactions) between the different directional components of the velocity field.

The Clebsch representation for the velocity field can be written as[5,6]

$$\vec{v}_\nu = f \ \partial_\nu q \tag{3}$$

and the related vorticity is

$$\vec{\omega}_{\mu\nu} = \partial_\mu \vec{v}_\nu - \partial_\nu \vec{v}_\mu \tag{4}$$

where

$$f = -1/[2\pi(1+\rho_c^2)] \tag{5}$$

ρ_c is a constant amplitude so f is also a constant. We consider ρ_c as an analogy with a constant amplitude of electromagnetic wave in a vacuum space[7], q is the phase. f and q are Clebsch variables, both are scalars.

The identical gauge potential and field strength tensor are

$$\vec{A}_\nu = f \ \partial_\nu q \tag{6}$$

and

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu \tag{7}$$

respectively.

.

We would like to thank Richard Tao Roni Hutagalung and AI (Chat GPT) for fruitful discussions.

Special thanks to beloved Juwita Armilia and Aliya Syauqina Hadi for much love. To Ibunda and Ayahanda, may Allah bless them with the Jannatul Firdaus.

This research is fully supported by self-funding.

[1] Roman Jackiw, *Lectures on Fluid Dynamics*, Springer, 2002.
[2] Y. M. Cho, Seung Hun Oh, Pengming Zhang, *Knots in Physics*, International Journal of Modern Physics A, Vol. 33, No. 07, 1830006 (2018).
[3] Yi-Shi Duan, Xin Liu, Li-bin Fu, *Many knots in Chern-Simons field theory*, Physical Review D **67**, 085022 (2003).
[4] A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, 2010.
[5] Miftachul Hadi, *Knot in weak-field geometrical optics*, OSF, https://osf.io/preprints/osf/e7afj, 2023, and all references therein.
[6] A. F. Ranada, A. Tiemblo, *A Topological Structure in the Set of Classical Free Radiation Electromagnetic Fields*, arXiv:1407.8145v1 [physics.class-ph] 29 Jul 2014.
[7] Edward M. Purcell, David J. Morin, *Electricity and Magnetism*, Cambridge University Press, 2013. Roald K. Wangsness, *Electromagnetic Fields*, John Wiley & Sons, 1986.