A Reformulation of Classical Mechanics

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In classical mechanics, a new reformulation is presented, which is invariant under transformations between inertial and non-inertial reference frames and which can be applied in any reference frame without introducing fictitious forces. Additionally, in this paper, we assume that all forces can obey or disobey Newton's third law.

Introduction

The new reformulation in classical mechanics presented in this paper is obtained starting from an auxiliary system of particles (called free-system) that is used to obtain kinematic magnitudes (for example, inertial position, inertial velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The inertial position $\mathbf{r}_i$, the inertial velocity $\mathbf{v}_i$ and the inertial acceleration $\mathbf{a}_i$ of a particle i relative to a reference frame S (inertial or non-inertial) are given by:

$$\mathbf{r}_i \doteq (\tilde{r}_i) = (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq d(\tilde{r}_i)/dt = (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq d^2(\tilde{r}_i)/dt^2 = (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

where $\tilde{r}_i$ is the position vector of particle i relative to the auxiliary frame [$\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system] [relative to the frame S] (see Annex I)

The auxiliary frame is a reference frame fixed to the free-system ($\vec{\omega} = 0$) whose origin always coincides with the center of mass of the free-system ($\vec{R} = \vec{V} = \vec{A} = 0$)

Any reference frame S is an inertial frame when the angular velocity $\vec{\omega}$ of the free-system and the acceleration $\vec{A}$ of the center of mass of the free-system are equal to zero relative to S.

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The net force $\mathbf{F}_i$ acting on a particle i of mass m_i produces an inertial acceleration $\mathbf{a}_i$ according to the following equation: [$\mathbf{F}_i = m_i\,\mathbf{a}_i$]

[3] In this paper, we assume that all forces can obey or disobey Newton's third law in its weak form or in its strong form.

The Equation of Motion

The net force $\mathbf{F}_i$ acting on a particle i of mass m_i produces an inertial acceleration $\mathbf{a}_i$ according to the following equation:

$$\mathbf{F}_i = m_i \, \mathbf{a}_i$$

From the above equation it follows that the (ordinary) acceleration $\vec{a}_i$ of particle i relative to a reference frame S (inertial or non-inertial) is given by:

$$\vec{a}_i = \mathbf{F}_i/m_i + \vec{A} + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system (see Annex I)

From the above equation it follows that particle i can have a non-zero acceleration even if there is no force acting on particle i, and also that particle i can have zero acceleration (state of rest or of uniform linear motion) even if there is an unbalanced net force acting on particle i.

However, from the above equation it also follows that Newton's first and second laws are valid in any inertial reference frame, since the angular velocity $\vec{\omega}$ of the free-system and the acceleration $\vec{A}$ of the center of mass of the free-system are equal to zero relative to any inertial reference frame.

In this paper, any reference frame S is an inertial frame when the angular velocity $\vec{\omega}$ of the free-system and the acceleration $\vec{A}$ of the center of mass of the free-system are equal to zero relative to the frame S. Therefore, any reference frame S is a non-inertial frame when the angular velocity $\vec{\omega}$ of the free-system or the acceleration $\vec{A}$ of the center of mass of the free-system are not equal to zero relative to the frame S.

However, since in classical mechanics any reference frame is actually an ideal rigid body then any reference frame S is an inertial frame when the net force acting at each point of the frame S is equal to zero. Therefore, any reference frame S is a non-inertial frame when the net force acting at each point of the frame S is not equal to zero (see Annex IV)

On the other hand, the new reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics.

However, non-inertial observers can use Newtonian mechanics only if they introduce fictitious forces into $\mathbf{F}_i$ (such as the centrifugal force, the Coriolis force, etc.)

Additionally, the new reformulation of classical mechanics presented in this paper is also a relational reformulation of classical mechanics since it is obtained starting from relative magnitudes (position, velocity and acceleration) between particles.

However, as already stated above, the new reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$
Position CM 1	$\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$
Velocity CM 1	$\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$
Acceleration CM 1	$\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$
Position CM 2	$\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$
Velocity CM 2	$\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Acceleration CM 2	$\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$
Linear Momentum 1	$\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Angular Momentum 1	$\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$
Angular Momentum 2	$\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$
Work 1	$\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \, \mathrm{K}_1$
Kinetic Energy 1	$\Delta \, \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$
Potential Energy 1	$\Delta \, \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$
Mechanical Energy 1	$\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$
Lagrangian 1	$\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$
Work 2	$\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \, \mathrm{K}_2$
Kinetic Energy 2	$\Delta \, \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$
Potential Energy 2	$\Delta \, \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 2	$\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$
Lagrangian 2	$\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3 $\quad \mathrm{W}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i = \Delta\, \mathrm{K}_3$

Kinetic Energy 3 $\quad \Delta\, \mathrm{K}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$

Potential Energy 3 $\quad \Delta\, \mathrm{U}_3 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$

Mechanical Energy 3 $\quad \mathrm{E}_3 \doteq \mathrm{K}_3 + \mathrm{U}_3$

Work 4 $\quad \mathrm{W}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta\, \mathrm{K}_4$

Kinetic Energy 4 $\quad \Delta\, \mathrm{K}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \right]$

Potential Energy 4 $\quad \Delta\, \mathrm{U}_4 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$

Mechanical Energy 4 $\quad \mathrm{E}_4 \doteq \mathrm{K}_4 + \mathrm{U}_4$

Work 5 $\quad \mathrm{W}_5 \doteq \sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right] = \Delta\, \mathrm{K}_5$

Kinetic Energy 5 $\quad \Delta\, \mathrm{K}_5 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R}) \right]$

Potential Energy 5 $\quad \Delta\, \mathrm{U}_5 \doteq -\sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right]$

Mechanical Energy 5 $\quad \mathrm{E}_5 \doteq \mathrm{K}_5 + \mathrm{U}_5$

Work 6 $\quad \mathrm{W}_6 \doteq \sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right] = \Delta\, \mathrm{K}_6$

Kinetic Energy 6 $\quad \Delta\, \mathrm{K}_6 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$

Potential Energy 6 $\quad \Delta\, \mathrm{U}_6 \doteq -\sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$

Mechanical Energy 6 $\quad \mathrm{E}_6 \doteq \mathrm{K}_6 + \mathrm{U}_6$

The Relations

From the above definitions, the following relations can be obtained (see Annex II)

$$\mathrm{K}_1 = \mathrm{K}_2 + {}^1\!/_2\, \mathrm{M}\, \mathbf{V}_{cm}^2$$

$$\mathrm{K}_3 = \mathrm{K}_4 + {}^1\!/_2\, \mathrm{M}\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$\mathrm{K}_5 = \mathrm{K}_6 + {}^1\!/_2\, \mathrm{M} \left[(\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R}) \right]$$

$$\mathrm{K}_5 = \mathrm{K}_1 + \mathrm{K}_3 \quad \& \quad \mathrm{U}_5 = \mathrm{U}_1 + \mathrm{U}_3 \quad \& \quad \mathrm{E}_5 = \mathrm{E}_1 + \mathrm{E}_3$$

$$\mathrm{K}_6 = \mathrm{K}_2 + \mathrm{K}_4 \quad \& \quad \mathrm{U}_6 = \mathrm{U}_2 + \mathrm{U}_4 \quad \& \quad \mathrm{E}_6 = \mathrm{E}_2 + \mathrm{E}_4$$

The Conservation Laws

The linear momentum $[\,\mathbf{P}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \;=\; \text{constant} \qquad \left[\, d(\mathbf{P}_1)/dt \;=\; \textstyle\sum_i^{\rm N} m_i\,\mathbf{a}_i \;=\; \textstyle\sum_i^{\rm N} \mathbf{F}_i \;=\; 0 \,\right]$$

The angular momentum $[\,\mathbf{L}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \;=\; \text{constant} \qquad \left[\, d(\mathbf{L}_1)/dt \;=\; \textstyle\sum_i^{\rm N} m_i \left[\, \mathbf{r}_i \times \mathbf{a}_i \,\right] \;=\; \textstyle\sum_i^{\rm N} \mathbf{r}_i \times \mathbf{F}_i \;=\; 0 \,\right]$$

The angular momentum $[\,\mathbf{L}_2\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \;=\; \text{constant} \qquad \left[\, d(\mathbf{L}_2)/dt \;=\; \textstyle\sum_i^{\rm N} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm}) \,\right] \;=\;\right.$$

$$\left. \textstyle\sum_i^{\rm N} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i \,\right] \;=\; \textstyle\sum_i^{\rm N} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \;=\; 0 \,\right]$$

The mechanical energy $[\,\mathrm{E}_1\,]$ and the mechanical energy $[\,\mathrm{E}_2\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_1 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_1 \;=\; \Delta\,\mathrm{K}_1 + \Delta\,\mathrm{U}_1 \;=\; 0 \,\right]$$

$$\mathrm{E}_2 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_2 \;=\; \Delta\,\mathrm{K}_2 + \Delta\,\mathrm{U}_2 \;=\; 0 \,\right]$$

The mechanical energy $[\,\mathrm{E}_3\,]$ and the mechanical energy $[\,\mathrm{E}_4\,]$ of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \;=\; \text{constant} \qquad \left[\, \mathrm{E}_3 \;=\; \textstyle\sum_i^{\rm N} 1/2 \left[\, m_i\,\mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i \,\right] \;=\; 0 \,\right]$$

$$\mathrm{E}_4 \;=\; \text{constant} \qquad \left[\, \mathrm{E}_4 \;=\; \textstyle\sum_i^{\rm N} 1/2 \left[\, m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \;=\; 0 \,\right]$$

$$\textstyle\sum_i^{\rm N} 1/2\; m_i \left[\, (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \;=\; \textstyle\sum_i^{\rm N} 1/2\; m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy $[\,\mathrm{E}_5\,]$ and the mechanical energy $[\,\mathrm{E}_6\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_5 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_5 \;=\; \Delta\,\mathrm{K}_5 + \Delta\,\mathrm{U}_5 \;=\; 0 \,\right]$$

$$\mathrm{E}_6 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_6 \;=\; \Delta\,\mathrm{K}_6 + \Delta\,\mathrm{U}_6 \;=\; 0 \,\right]$$

General Observations

All the equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Therefore, the new reformulation of classical mechanics presented in this paper is totally in accordance with the general principle of relativity.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$ (such as the centrifugal force, the Coriolis force, etc.)

In this paper, the magnitudes [m, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, M, $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\mathbf{F}$, $\mathbf{P}_1$, $\mathbf{L}_1$, $\mathbf{L}_2$, W_1, K_1, U_1, E_1, L_1, W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and E_6] are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero [$E_3 = K_3 + U_3 = 0$]

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles [$E_5 = E_1$]

The mechanical energy E_4 of a system of particles is always zero [$E_4 = K_4 + U_4 = 0$]

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles [$E_6 = E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: [$U_3 = (\frac{k}{2})\, U_1$] and [$U_5 = (1+\frac{k}{2})\, U_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: [$U_4 = (\frac{k}{2})\, U_2$] and [$U_6 = (1+\frac{k}{2})\, U_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: [$K_1 = -\, K_3 = U_3 = (\frac{k}{2})\, U_1 = (\frac{k}{2+k})\, E_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: [$K_2 = -\, K_4 = U_4 = (\frac{k}{2})\, U_2 = (\frac{k}{2+k})\, E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_5 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_1 \rangle = -\, \langle K_3 \rangle = \langle U_3 \rangle = (\frac{k}{2})\, \langle U_1 \rangle = (\frac{k}{2+k})\, \langle E_1 \rangle$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_6 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_2 \rangle = -\, \langle K_4 \rangle = \langle U_4 \rangle = (\frac{k}{2})\, \langle U_2 \rangle = (\frac{k}{2+k})\, \langle E_2 \rangle$]

The average kinetic energy $\langle K_5 \rangle$ and the average kinetic energy $\langle K_6 \rangle$ of a system of particles with bounded motion are related to the virial theorem.

The average kinetic energy $\langle \mathrm{K}_5 \rangle$ and the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of a system of particles with bounded motion (in $\langle \mathrm{K}_5 \rangle$ relative to $\vec{R}$ and in $\langle \mathrm{K}_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\dot{r}_i\,\dot{r}_i + \ddot{r}_i\,r_i\,)\ ]$ where $r_i \doteq |\,\vec{r}_i - \vec{R}\,|$ and $[\ \mathrm{K}_6 = \sum_{i<j}^{\mathrm{N}} 1/2\, m_i\, m_j\, \mathrm{M}^{-1}(\,\dot{r}_{ij}\,\dot{r}_{ij} + \ddot{r}_{ij}\,r_{ij}\,)\ ]$ where $r_{ij} \doteq |\,\vec{r}_i - \vec{r}_j\,|$ Note 1 $\left(\sum_{i<j}^{\mathrm{N}} \doteq \sum_{i=1}^{\mathrm{N}} \sum_{j>i}^{\mathrm{N}}\right)$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\ddot{\tau}_i\,)\ ]$ where $\tau_i \doteq 1/2\ (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and $[\ \mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2\, m_i\, m_j\, \mathrm{M}^{-1}(\,\ddot{\tau}_{ij}\,)\ ]$ where $\tau_{ij} \doteq 1/2\ (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$ Note 2 $\left(\sum_{j>i}^{\mathrm{N}} \doteq \sum_{i=1}^{\mathrm{N}} \sum_{j>i}^{\mathrm{N}}\right)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the free-system [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal forces obey Newton's third law in its weak form $[\,\mathrm{U}_2 = \mathrm{U}_1\,]$

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal forces obey Newton's third law in its weak form $[\,\mathrm{U}_4 = \mathrm{U}_3\,]$

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal forces obey Newton's third law in its weak form $[\,\mathrm{U}_6 = \mathrm{U}_5\,]$

A reference frame S is a special non-rotating frame when the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also an inertial frame when the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

If the origin of a special non-rotating frame S $[\,\vec{\omega} = 0\,]$ always coincides with the center of mass of the free-system $[\,\vec{R} = \vec{V} = \vec{A} = 0\,]$ then relative to S: $[\,\mathbf{r}_i = \vec{r}_i$, $\mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i\,]$ Therefore, it is easy to see that inertial magnitudes and ordinary magnitudes are always the same in the reference frame S.

This paper does not contradict Newton's first and second laws since these two laws are valid in all inertial reference frames. The equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ is a simple reformulation of Newton's second law.

Finally, in this paper, the equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ is valid in all reference frames (inertial or non-inertial) even if all forces always disobey Newton's third law in its strong form and in its weak form.

Bibliography

A. Blato, A Reformulation of Classical Mechanics.

A. Blatter, A Reformulation of Classical Mechanics.

A. Torassa, A Reformulation of Classical Mechanics.

Annex I

The Free-System

The free-system is a system of N particles that must always be free of internal and external forces, that must be three-dimensional, and that the relative distances between the N particles must be constant.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the free-system relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the free-system relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{\mathrm{N}} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i\, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i\, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i\, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \,[\, |\, \vec{r}_i - \vec{R}\, |^2\, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{\mathrm{N}} m_i\, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the free-system, $\overleftrightarrow{I}$ is the inertia tensor of the free-system (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the free-system relative to the reference frame S.

The Transformations

The transformations of position, velocity and acceleration of a particle i between a reference frame S and another reference frame S', are given by:

$$(\vec{r}_i - \vec{R}) = \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') = \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) = \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') = \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) = \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2\, \vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\, \vec{\omega}' \times (\vec{r}'_i - \vec{R}')\,] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') = \mathbf{a}'_i = \mathbf{a}_i$$

Annex II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i = (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} = (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i = (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} = (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) = (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) = \big[\, (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\, (\vec{v}_i - \vec{V}_{cm}) \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\, (\vec{r}_i - \vec{R}_{cm}) \cdot \big[\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \big\{\, (\vec{a}_i - \vec{A}_{cm}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,] -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) +$$

$$\big\{\, \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,] \,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\, \vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) -$$

$$\big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{\, \big[\, \vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \,\big]\, \vec{\omega} - (\, \vec{\omega} \cdot \vec{\omega} \,)\, (\vec{r}_i - \vec{R}_{cm}) \,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) =$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\, \vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \,\big]^2 - (\, \vec{\omega} \,)^2\, (\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Annex III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j)\, \big]$$

$$\mathrm{W}_2 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$\Delta\, \mathrm{K}_2 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, m_j\, \mathrm{M}^{-1} \,(\mathbf{v}_i - \mathbf{v}_j)^2 \;=\; \mathrm{W}_2$$

$$\Delta\, \mathrm{U}_2 \;=\; -\textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$\mathrm{W}_4 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$\Delta\, \mathrm{K}_4 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)\, \big] \;=\; \mathrm{W}_4$$

$$\Delta\, \mathrm{U}_4 \;=\; -\textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$\mathrm{W}_6 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j)\, \big]$$

$$\Delta\, \mathrm{K}_6 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j)\, \big] \;=\; \mathrm{W}_6$$

$$\Delta\, \mathrm{U}_6 \;=\; -\textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j)\, \big]$$

The magnitudes $\mathrm{W}_{(1 \text{ to } 6)}$ and $\mathrm{U}_{(1 \text{ to } 6)}$ of an isolated system of N particles, whose internal forces obey Newton's third law in its weak form, can be reduced to:

$$\mathrm{W}_1 \;=\; \mathrm{W}_2 \;=\; \textstyle\sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta\, \mathrm{U}_1 \;=\; \Delta\, \mathrm{U}_2 \;=\; -\textstyle\sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\mathrm{W}_3 \;=\; \mathrm{W}_4 \;=\; \textstyle\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta\, \mathrm{U}_3 \;=\; \Delta\, \mathrm{U}_4 \;=\; -\textstyle\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\mathrm{W}_5 \;=\; \mathrm{W}_6 \;=\; \textstyle\sum_i^{\mathrm{N}} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i\, \big]$$

$$\Delta\, \mathrm{U}_5 \;=\; \Delta\, \mathrm{U}_6 \;=\; -\textstyle\sum_i^{\mathrm{N}} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i\, \big]$$

Annex IV

Frames and Forces

Diagram of net forces acting on a reference frame S, when the reference frame S is a linearly non-accelerated and non-rotating frame relative to an inertial frame (9 points)

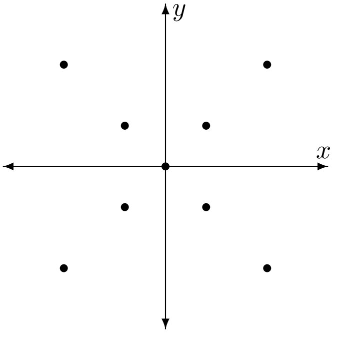


Diagram of net forces acting on a reference frame S, when the reference frame S is a linearly accelerated and non-rotating frame relative to an inertial frame (9 points)

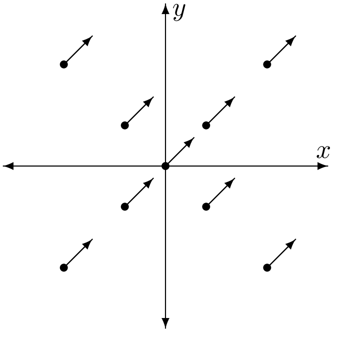
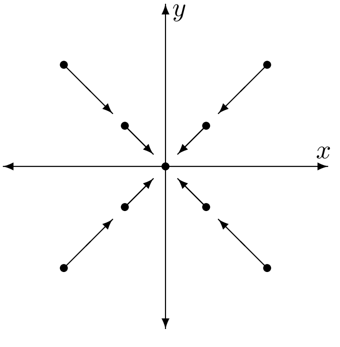


Diagram of net forces acting on a reference frame S, when the reference frame S is a linearly non-accelerated and rotating frame relative to an inertial frame (9 points)



A Reformulation of Classical Mechanics

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(Paper IV)

In classical mechanics, a new reformulation is presented, which is invariant under transformations between inertial and non-inertial reference frames, which can be applied in any reference frame without introducing fictitious forces and which establishes the existence of new universal forces of interaction, called kinetic forces.

Introduction

The new reformulation in classical mechanics presented in this paper is obtained starting from an auxiliary system of particles (called free-system) that is used to obtain kinematic magnitudes (for example, inertial position, inertial velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The inertial position $\mathbf{r}_i$, the inertial velocity $\mathbf{v}_i$ and the inertial acceleration $\mathbf{a}_i$ of a particle i relative to a reference frame S (inertial or non-inertial) are given by:

$$\mathbf{r}_i \doteq (\tilde{r}_i) = (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq d(\tilde{r}_i)/dt = (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq d^2(\tilde{r}_i)/dt^2 = (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

where $\tilde{r}_i$ is the position vector of particle i relative to the auxiliary frame [$\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system] [relative to the frame S] (see Annex I)

The auxiliary frame is a reference frame fixed to the free-system ($\vec{\omega} = 0$) whose origin always coincides with the center of mass of the free-system ($\vec{R} = \vec{V} = \vec{A} = 0$)

Any reference frame S is an inertial frame when the angular velocity $\vec{\omega}$ of the free-system and the acceleration $\vec{A}$ of the center of mass of the free-system are equal to zero relative to S.

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The total force $\mathbf{T}_i$ acting on a particle i is always zero [$\mathbf{T}_i = 0$]

[3] In this paper, we assume that all dynamic forces (all non-kinetic forces) can obey or disobey Newton's third law in its weak form or in its strong form.

The Kinetic Forces

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j, caused by the interaction between particle i and particle j, is given by:

$$\mathbf{K}^a_{ij} \;=\; -\,\frac{m_i\,m_j}{M}\,(\mathbf{a}_i - \mathbf{a}_j)$$

where $\mathbf{a}_i$ is the inertial acceleration of particle i, $\mathbf{a}_j$ is the inertial acceleration of particle j, and M $(= \sum_i^{All}\, m_i\,)$ is the mass of the Universe.

The kinetic force $\mathbf{K}^u_i$ exerted on a particle i of mass m_i by the center of mass of the Universe, caused by the interaction between particle i and the Universe, is given by:

$$\mathbf{K}^u_i \;=\; -\,m_i\,\mathbf{A}_{cm}$$

where $\mathbf{A}_{cm}$ $(=\, M^{-1}\sum_i^{All}\, m_i\,\mathbf{a}_i\,)$ is the inertial acceleration of the center of mass of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ $(= \sum_j^{All}\, \mathbf{K}^a_{ij} + \mathbf{K}^u_i\,)$ acting on a particle i of mass m_i is given by:

$$\mathbf{K}_i \;=\; -\,m_i\,\mathbf{a}_i$$

where $\mathbf{a}_i$ is the inertial acceleration of particle i.

If all dynamic forces always obey Newton's third law in its weak form then the inertial acceleration of the center of mass of the Universe $\mathbf{A}_{cm}$ is always zero.

On the other hand, the kinetic force $\mathbf{K}^a$ always obey Newton's third law in its weak form or in its strong form.

The [2] Principle

The second principle of the new dynamics establishes that the total force $\mathbf{T}_i$ acting on a particle i is always zero.

$$\mathbf{T}_i \;=\; 0$$

If the total force $\mathbf{T}_i$ is divided into the following two parts: the net kinetic force $\mathbf{K}_i$ and the net dynamic force $\mathbf{F}_i$ ($\sum$ of gravitational forces, electrostatic forces, etc.) then we have:

$$\mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Now, substituting $\mathbf{K}_i$ $(= -\,m_i\,\mathbf{a}_i\,)$ and rearranging, we finally obtain:

$$\mathbf{F}_i \;=\; m_i\,\mathbf{a}_i$$

This equation (similar to Newton's second law) will be used throughout this paper.

On the other hand, in this paper a system of particles is isolated when the system is free of external dynamic forces.

The Equation of Motion

The net dynamic force $\mathbf{F}_i$ acting on a particle i of mass m_i is related to the inertial acceleration $\mathbf{a}_i$ of particle i according to the following equation:

$$\mathbf{F}_i = m_i \, \mathbf{a}_i$$

From the above equation it follows that the (ordinary) acceleration $\vec{a}_i$ of particle i relative to a reference frame S (inertial or non-inertial) is given by:

$$\vec{a}_i = \mathbf{F}_i/m_i + \vec{A} + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system (see Annex I)

From the above equation it follows that particle i can have a non-zero acceleration even if there is no dynamic force acting on particle i, and also that particle i can have zero acceleration (state of rest or of uniform linear motion) even if there is an unbalanced net dynamic force acting on particle i.

However, from the above equation it also follows that Newton's first and second laws are valid in any inertial reference frame, since the angular velocity $\vec{\omega}$ of the free-system and the acceleration $\vec{A}$ of the center of mass of the free-system are equal to zero relative to any inertial reference frame.

In this paper, any reference frame S is an inertial frame when the angular velocity $\vec{\omega}$ of the free-system and the acceleration $\vec{A}$ of the center of mass of the free-system are equal to zero relative to the frame S. Therefore, any reference frame S is a non-inertial frame when the angular velocity $\vec{\omega}$ of the free-system or the acceleration $\vec{A}$ of the center of mass of the free-system are not equal to zero relative to the frame S.

However, since in classical mechanics any reference frame is actually an ideal rigid body then any reference frame S is an inertial frame when the net dynamic force acting at each point of the frame S is equal to zero. Therefore, any reference frame S is a non-inertial frame when the net dynamic force acting at each point of the frame S is not equal to zero (see Annex IV)

On the other hand, the new reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics.

However, non-inertial observers can use Newtonian mechanics only if they introduce fictitious forces into $\mathbf{F}_i$ (such as the centrifugal force, the Coriolis force, etc.)

Additionally, the new reformulation of classical mechanics presented in this paper is also a relational reformulation of classical mechanics since it is obtained starting from relative magnitudes (position, velocity and acceleration) between particles.

However, as already stated above, the new reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$
Position CM 1	$\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$
Velocity CM 1	$\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$
Acceleration CM 1	$\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$
Position CM 2	$\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$
Velocity CM 2	$\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Acceleration CM 2	$\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$
Linear Momentum 1	$\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Angular Momentum 1	$\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$
Angular Momentum 2	$\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$
Work 1	$\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \mathrm{K}_1$
Kinetic Energy 1	$\Delta \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$
Potential Energy 1	$\Delta \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$
Mechanical Energy 1	$\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$
Lagrangian 1	$\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$
Work 2	$\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \mathrm{K}_2$
Kinetic Energy 2	$\Delta \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$
Potential Energy 2	$\Delta \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 2	$\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$
Lagrangian 2	$\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3	$W_3 \doteq \sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i = \Delta\, K_3$
Kinetic Energy 3	$\Delta\, K_3 \doteq \sum_i^N \Delta\, {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$
Potential Energy 3	$\Delta\, U_3 \doteq -\sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$
Mechanical Energy 3	$E_3 \doteq K_3 + U_3$
Work 4	$W_4 \doteq \sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta\, K_4$
Kinetic Energy 4	$\Delta\, K_4 \doteq \sum_i^N \Delta\, {}^1\!/_2\, m_i \left[(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \right]$
Potential Energy 4	$\Delta\, U_4 \doteq -\sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 4	$E_4 \doteq K_4 + U_4$
Work 5	$W_5 \doteq \sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right] = \Delta\, K_5$
Kinetic Energy 5	$\Delta\, K_5 \doteq \sum_i^N \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R}) \right]$
Potential Energy 5	$\Delta\, U_5 \doteq -\sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right]$
Mechanical Energy 5	$E_5 \doteq K_5 + U_5$
Work 6	$W_6 \doteq \sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right] = \Delta\, K_6$
Kinetic Energy 6	$\Delta\, K_6 \doteq \sum_i^N \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$
Potential Energy 6	$\Delta\, U_6 \doteq -\sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$
Mechanical Energy 6	$E_6 \doteq K_6 + U_6$

The Relations

From the above definitions, the following relations can be obtained (see Annex II)

$$K_1 = K_2 + {}^1\!/_2\, M\, \mathbf{V}_{cm}^2$$

$$K_3 = K_4 + {}^1\!/_2\, M\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$K_5 = K_6 + {}^1\!/_2\, M \left[(\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R}) \right]$$

$$K_5 = K_1 + K_3 \quad \& \quad U_5 = U_1 + U_3 \quad \& \quad E_5 = E_1 + E_3$$

$$K_6 = K_2 + K_4 \quad \& \quad U_6 = U_2 + U_4 \quad \& \quad E_6 = E_2 + E_4$$

The Conservation Laws

The linear momentum [$\mathbf{P}_1$] of an isolated system of N particles remains constant if the internal dynamic forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \;=\; \text{constant} \qquad \left[\, d(\mathbf{P}_1)/dt \;=\; \textstyle\sum_i^{\text{N}} m_i\,\mathbf{a}_i \;=\; \sum_i^{\text{N}} \mathbf{F}_i \;=\; 0 \,\right]$$

The angular momentum [$\mathbf{L}_1$] of an isolated system of N particles remains constant if the internal dynamic forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \;=\; \text{constant} \qquad \left[\, d(\mathbf{L}_1)/dt \;=\; \textstyle\sum_i^{\text{N}} m_i \left[\, \mathbf{r}_i \times \mathbf{a}_i \,\right] \;=\; \sum_i^{\text{N}} \mathbf{r}_i \times \mathbf{F}_i \;=\; 0 \,\right]$$

The angular momentum [$\mathbf{L}_2$] of an isolated system of N particles remains constant if the internal dynamic forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \;=\; \text{constant} \qquad \left[\, d(\mathbf{L}_2)/dt \;=\; \textstyle\sum_i^{\text{N}} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm}) \,\right] \;=\right.$$

$$\left. \textstyle\sum_i^{\text{N}} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i \,\right] \;=\; \sum_i^{\text{N}} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \;=\; 0 \,\right]$$

The mechanical energy [E_1] and the mechanical energy [E_2] of a system of N particles remain constant if the system is only subject to kinetic forces and to conservative dynamic forces.

$$\mathrm{E}_1 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_1 \;=\; \Delta\,\mathrm{K}_1 + \Delta\,\mathrm{U}_1 \;=\; 0 \,\right]$$

$$\mathrm{E}_2 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_2 \;=\; \Delta\,\mathrm{K}_2 + \Delta\,\mathrm{U}_2 \;=\; 0 \,\right]$$

The mechanical energy [E_3] and the mechanical energy [E_4] of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \;=\; \text{constant} \qquad \left[\, \mathrm{E}_3 \;=\; \textstyle\sum_i^{\text{N}} 1/2 \left[\, m_i\,\mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i \,\right] \;=\; 0 \,\right]$$

$$\mathrm{E}_4 \;=\; \text{constant} \qquad \left[\, \mathrm{E}_4 \;=\; \textstyle\sum_i^{\text{N}} 1/2 \left[\, m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \;=\; 0 \,\right]$$

$$\textstyle\sum_i^{\text{N}} 1/2\, m_i \left[\, (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \;=\; \sum_i^{\text{N}} 1/2\, m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy [E_5] and the mechanical energy [E_6] of a system of N particles remain constant if the system is only subject to kinetic forces and to conservative dynamic forces.

$$\mathrm{E}_5 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_5 \;=\; \Delta\,\mathrm{K}_5 + \Delta\,\mathrm{U}_5 \;=\; 0 \,\right]$$

$$\mathrm{E}_6 \;=\; \text{constant} \qquad \left[\, \Delta\,\mathrm{E}_6 \;=\; \Delta\,\mathrm{K}_6 + \Delta\,\mathrm{U}_6 \;=\; 0 \,\right]$$

General Observations

All the equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Therefore, the new reformulation of classical mechanics presented in this paper is totally in accordance with the general principle of relativity.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$ (such as the centrifugal force, the Coriolis force, etc.)

In this paper, the magnitudes [$m, \mathbf{r}, \mathbf{v}, \mathbf{a}, \mathrm{M}, \mathbf{R}, \mathbf{V}, \mathbf{A}, \mathbf{T}, \mathbf{K}, \mathbf{F}, \mathbf{P}_1, \mathbf{L}_1, \mathbf{L}_2, \mathrm{W}_1, \mathrm{K}_1, \mathrm{U}_1, \mathrm{E}_1, \mathrm{L}_1$ W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and E_6] are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero [$\mathrm{E}_3 = \mathrm{K}_3 + \mathrm{U}_3 = 0$]

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles [$\mathrm{E}_5 = \mathrm{E}_1$]

The mechanical energy E_4 of a system of particles is always zero [$\mathrm{E}_4 = \mathrm{K}_4 + \mathrm{U}_4 = 0$]

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles [$\mathrm{E}_6 = \mathrm{E}_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: [$\mathrm{U}_3 = (\frac{k}{2})\,\mathrm{U}_1$] and [$\mathrm{U}_5 = (1+\frac{k}{2})\,\mathrm{U}_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: [$\mathrm{U}_4 = (\frac{k}{2})\,\mathrm{U}_2$] and [$\mathrm{U}_6 = (1+\frac{k}{2})\,\mathrm{U}_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: [$\mathrm{K}_1 = -\,\mathrm{K}_3 = \mathrm{U}_3 = (\frac{k}{2})\,\mathrm{U}_1 = (\frac{k}{2+k})\,\mathrm{E}_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: [$\mathrm{K}_2 = -\,\mathrm{K}_4 = \mathrm{U}_4 = (\frac{k}{2})\,\mathrm{U}_2 = (\frac{k}{2+k})\,\mathrm{E}_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle \mathrm{K}_5 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle \mathrm{K}_1 \rangle = -\,\langle \mathrm{K}_3 \rangle = \langle \mathrm{U}_3 \rangle = (\frac{k}{2})\,\langle \mathrm{U}_1 \rangle = (\frac{k}{2+k})\,\langle \mathrm{E}_1 \rangle$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle \mathrm{K}_2 \rangle = -\,\langle \mathrm{K}_4 \rangle = \langle \mathrm{U}_4 \rangle = (\frac{k}{2})\,\langle \mathrm{U}_2 \rangle = (\frac{k}{2+k})\,\langle \mathrm{E}_2 \rangle$]

The average kinetic energy $\langle \mathrm{K}_5 \rangle$ and the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of a system of particles with bounded motion are related to the virial theorem.

The average kinetic energy $\langle \mathrm{K}_5 \rangle$ and the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of a system of particles with bounded motion (in $\langle \mathrm{K}_5 \rangle$ relative to $\vec{R}$ and in $\langle \mathrm{K}_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\dot{r}_i\,\dot{r}_i + \ddot{r}_i\,r_i\,)\ ]$ where $r_i \doteq |\,\vec{r}_i - \vec{R}\,|$ and $[\ \mathrm{K}_6 = \sum_{i<j}^{\mathrm{N}} 1/2\, m_i\, m_j\, \mathrm{M}^{-1}(\,\dot{r}_{ij}\,\dot{r}_{ij} + \ddot{r}_{ij}\,r_{ij}\,)\ ]$ where $r_{ij} \doteq |\,\vec{r}_i - \vec{r}_j\,|$ Note 1 $\left(\sum_{i<j}^{\mathrm{N}} \doteq \sum_{i=1}^{\mathrm{N}} \sum_{j>i}^{\mathrm{N}}\right)$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\ddot{\tau}_i\,)\ ]$ where $\tau_i \doteq 1/2\ (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and $[\ \mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2\, m_i\, m_j\, \mathrm{M}^{-1}(\,\ddot{\tau}_{ij}\,)\ ]$ where $\tau_{ij} \doteq 1/2\ (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$ Note 2 $\left(\sum_{j>i}^{\mathrm{N}} \doteq \sum_{i=1}^{\mathrm{N}} \sum_{j>i}^{\mathrm{N}}\right)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the free-system [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal dynamic forces obey Newton's third law in its weak form $[\,\mathrm{U}_2 = \mathrm{U}_1\,]$

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal dynamic forces obey Newton's third law in its weak form $[\,\mathrm{U}_4 = \mathrm{U}_3\,]$

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal dynamic forces obey Newton's third law in its weak form $[\,\mathrm{U}_6 = \mathrm{U}_5\,]$

A reference frame S is a special non-rotating frame when the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also an inertial frame when the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

If the origin of a special non-rotating frame S $[\,\vec{\omega} = 0\,]$ always coincides with the center of mass of the free-system $[\,\vec{R} = \vec{V} = \vec{A} = 0\,]$ then relative to S: $[\,\mathbf{r}_i = \vec{r}_i,\ \mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i\,]$ Therefore, it is easy to see that inertial magnitudes and ordinary magnitudes are always the same in the reference frame S.

If kinetic forces are excluded, then this paper does not contradict Newton's first and second laws since they are valid in all inertial reference frames. The equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ is a simple reformulation of Newton's second law.

Finally, in this paper, the equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ is valid in all reference frames (inertial or non-inertial) even if all dynamic forces always disobey Newton's third law in its strong form and in its weak form.

Bibliography

A. Blato, A Reformulation of Classical Mechanics.

A. Blatter, A Reformulation of Classical Mechanics.

A. Torassa, A Reformulation of Classical Mechanics.

Annex I

The Free-System

The free-system is a system of N particles that must always be free of internal and external dynamic forces, that must be three-dimensional, and that the relative distances between the N particles must be constant.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the free-system relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the free-system relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{\mathrm{N}} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i\, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i\, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i\, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \,[\,|\,\vec{r}_i - \vec{R}\,|^2\, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{\mathrm{N}} m_i\, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the free-system, $\overleftrightarrow{I}$ is the inertia tensor of the free-system (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the free-system relative to the reference frame S.

The Transformations

The transformations of position, velocity and acceleration of a particle i between a reference frame S and another reference frame S', are given by:

$$(\vec{r}_i - \vec{R}) \;=\; \mathbf{r}_i \;=\; \mathbf{r}'_i$$

$$(\vec{r}\,'_i - \vec{R}') \;=\; \mathbf{r}'_i \;=\; \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \;=\; \mathbf{v}_i \;=\; \mathbf{v}'_i$$

$$(\vec{v}\,'_i - \vec{V}') - \vec{\omega}\,' \times (\vec{r}\,'_i - \vec{R}') \;=\; \mathbf{v}'_i \;=\; \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \;=\; \mathbf{a}_i \;=\; \mathbf{a}'_i$$

$$(\vec{a}\,'_i - \vec{A}') - 2\, \vec{\omega}\,' \times (\vec{v}\,'_i - \vec{V}') + \vec{\omega}\,' \times [\,\vec{\omega}\,' \times (\vec{r}\,'_i - \vec{R}')\,] - \vec{\alpha}\,' \times (\vec{r}\,'_i - \vec{R}') \;=\; \mathbf{a}'_i \;=\; \mathbf{a}_i$$

Annex II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i \,=\, (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} \,=\, (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) \,=\, (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i \,=\, (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} \,=\, (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) \,=\, (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) \,=\, \big[\,(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \cdot \big[\,(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \,=$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\,(\vec{v}_i - \vec{V}_{cm}) \cdot \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] + \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \cdot \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \,=$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\,(\vec{r}_i - \vec{R}_{cm}) \cdot \big[\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm})\,\big] + \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \cdot \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \,=$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[\,2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm})\,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \cdot \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \,=$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[\,2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm})\,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,\big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,=\, \big\{\,(\vec{a}_i - \vec{A}_{cm}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,] -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm})\,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) \,=\, (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\,2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm})\,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) \,+$$

$$\big\{\,\vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})\,]\,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\,\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm})\,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) \,=\, (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \,-$$

$$\big[\,2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm})\,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{\,\big[\,\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm})\,\big]\,\vec{\omega} - (\,\vec{\omega} \cdot \vec{\omega}\,)\,(\vec{r}_i - \vec{R}_{cm})\,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) \,=$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\,2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm})\,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\,\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm})\,\big]^2 - (\,\vec{\omega}\,)^2\,(\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,=\, (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Annex III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j) \,\big]$$

$$\mathrm{W}_2 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta\, \mathrm{K}_2 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, m_i\, m_j\, \mathrm{M}^{-1}\, (\mathbf{v}_i - \mathbf{v}_j)^2 \;=\; \mathrm{W}_2$$

$$\Delta\, \mathrm{U}_2 \;=\; -\, \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\mathrm{W}_4 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta\, \mathrm{K}_4 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big] \;=\; \mathrm{W}_4$$

$$\Delta\, \mathrm{U}_4 \;=\; -\, \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\mathrm{W}_6 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/\!_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

$$\Delta\, \mathrm{K}_6 \;=\; \textstyle\sum_{j>i}^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, m_i\, m_j\, \mathrm{M}^{-1} \big[\, (\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big] \;=\; \mathrm{W}_6$$

$$\Delta\, \mathrm{U}_6 \;=\; -\, \textstyle\sum_{j>i}^{\mathrm{N}} m_i\, m_j\, \mathrm{M}^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/\!_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

The magnitudes $\mathrm{W}_{(1\text{ to }6)}$ and $\mathrm{U}_{(1\text{ to }6)}$ of an isolated system of N particles, whose internal dynamic forces obey Newton's third law in its weak form, can be reduced to:

$$\mathrm{W}_1 \;=\; \mathrm{W}_2 \;=\; \textstyle\sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta\, \mathrm{U}_1 \;=\; \Delta\, \mathrm{U}_2 \;=\; -\, \textstyle\sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\mathrm{W}_3 \;=\; \mathrm{W}_4 \;=\; \textstyle\sum_i^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta\, \mathrm{U}_3 \;=\; \Delta\, \mathrm{U}_4 \;=\; -\, \textstyle\sum_i^{\mathrm{N}} \Delta\, {}^1\!/\!_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\mathrm{W}_5 \;=\; \mathrm{W}_6 \;=\; \textstyle\sum_i^{\mathrm{N}} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/\!_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$

$$\Delta\, \mathrm{U}_5 \;=\; \Delta\, \mathrm{U}_6 \;=\; -\, \textstyle\sum_i^{\mathrm{N}} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/\!_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$

Annex IV

Frames and Forces

Diagram of net dynamic forces acting on a reference frame S, when the reference frame S is a linearly non-accelerated and non-rotating frame relative to an inertial frame (9 points)

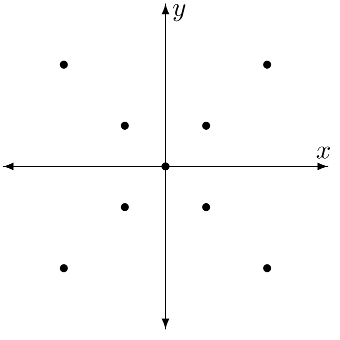


Diagram of net dynamic forces acting on a reference frame S, when the reference frame S is a linearly accelerated and non-rotating frame relative to an inertial frame (9 points)

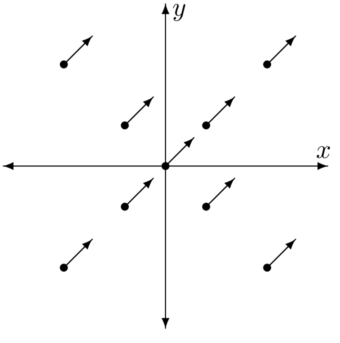
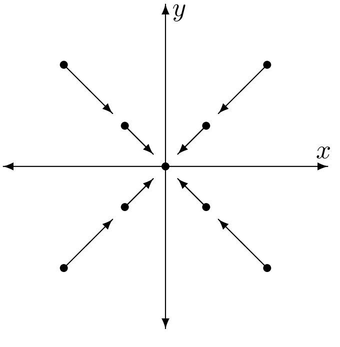


Diagram of net dynamic forces acting on a reference frame S, when the reference frame S is a linearly non-accelerated and rotating frame relative to an inertial frame (9 points)



Appendix A

Fields and Potentials I

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be expressed as follows:

$$\mathbf{K}_i \;=\; +\,m_i \Big[\,\mathbf{E} + (\vec{v}_i - \vec{V}\,) \times \mathbf{B}\,\Big]$$

$$\mathbf{K}_i \;=\; +\,m_i \Big[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{v}_i - \vec{V}\,) \times (\nabla \times \mathbf{A})\Big]$$

$$\mathbf{K}_i \;=\; +\,m_i \Big[-(\vec{a}_i - \vec{A}\,) + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}\,) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R}\,)\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R}\,)\Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^1\!/\!_2\,[\,\vec{\omega} \times (\vec{r}_i - \vec{R}\,)\,]^2 \;+\; {}^1\!/\!_2\,(\vec{v}_i - \vec{V}\,)^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{r}_i - \vec{R}\,)\,] + (\vec{v}_i - \vec{V}\,)$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; -\,\vec{\alpha} \times (\vec{r}_i - \vec{R}\,) + (\vec{a}_i - \vec{A}\,)\ *$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R}\,)\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be obtained starting from the following kinetic energy:

$$K_i \;=\; -\,m_i\,[\,\phi - (\vec{v}_i - \vec{V}\,) \cdot \mathbf{A}\,]$$

$$K_i \;=\; {}^1\!/\!_2\,m_i\,\big[\,(\vec{v}_i - \vec{V}\,) - \vec{\omega} \times (\vec{r}_i - \vec{R}\,)\,\big]^2$$

$$K_i \;=\; {}^1\!/\!_2\,m_i\,\big[\,\mathbf{v}_i\,\big]^2$$

Since the kinetic energy K_i must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i \;=\; -\,\frac{d}{dt}\left[\frac{\partial\,{}^1\!/\!_2\,m_i\,\big[\,\mathbf{v}_i\,\big]^2}{\partial\,\mathbf{v}_i}\right] + \frac{\partial\,{}^1\!/\!_2\,m_i\,\big[\,\mathbf{v}_i\,\big]^2}{\partial\,\mathbf{r}_i} \;=\; -\,m_i\,\mathbf{a}_i$$

where $\mathbf{r}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ are the inertial position, the inertial velocity and the inertial acceleration of particle i.

* In the temporal partial derivative, the spatial coordinates must be treated as constants [or replace this in the first equation: $+\,{}^1\!/\!_2\,(\vec{v}_i - \vec{V}\,) \times \mathbf{B}$, and this in the second equation: $+\,{}^1\!/\!_2\,(\vec{v}_i - \vec{V}\,) \times (\nabla \times \mathbf{A})$]

Appendix B

Fields and Potentials II

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i (relative to a reference frame S fixed to a particle s ($\vec{r}_s = \vec{v}_s = \vec{a}_s = 0$) of mass m_s, with inertial velocity $\mathbf{v}_s$ and inertial acceleration $\mathbf{a}_s$) can also be expressed as follows:

$$\mathbf{K}_i \;=\; +\, m_i \Big[\, \mathbf{E} + \vec{v}_i \times \mathbf{B} \,\Big]$$

$$\mathbf{K}_i \;=\; +\, m_i \Big[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \vec{v}_i \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_i \;=\; +\, m_i \Big[-(\vec{a}_i + \mathbf{a}_s) + 2\,\vec{\omega} \times \vec{v}_i - \vec{\omega} \times (\vec{\omega} \times \vec{r}_i\,) + \vec{\alpha} \times \vec{r}_i \,\Big]$$

where:

$$\mathbf{E} \;=\; -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^1\!/_2\,(\vec{\omega} \times \vec{r}_i\,)^2 \;+\; {}^1\!/_2\,(\vec{v}_i + \mathbf{v}_s)^2$$

$$\mathbf{A} \;=\; -\,(\vec{\omega} \times \vec{r}_i\,) + (\vec{v}_i + \mathbf{v}_s)$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; -\,\vec{\alpha} \times \vec{r}_i \,+ (\vec{a}_i + \mathbf{a}_s)\ ^*$$

$$\nabla\phi \;=\; \vec{\omega} \times (\vec{\omega} \times \vec{r}_i\,)$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be obtained starting from the following kinetic energy:

$$K_i \;=\; -\, m_i \,[\, \phi - (\vec{v}_i + \mathbf{v}_s) \cdot \mathbf{A} \,]$$

$$K_i \;=\; {}^1\!/_2\, m_i \,\big[\, (\vec{v}_i + \mathbf{v}_s) - (\vec{\omega} \times \vec{r}_i\,) \,\big]^2$$

$$K_i \;=\; {}^1\!/_2\, m_i \,\big[\, \mathbf{v}_i \,\big]^2$$

Since the kinetic energy K_i must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i \;=\; -\,\frac{d}{dt}\left[\, \frac{\partial\, {}^1\!/_2\, m_i \,[\, \mathbf{v}_i \,]^2}{\partial\, \mathbf{v}_i} \,\right] + \frac{\partial\, {}^1\!/_2\, m_i \,[\, \mathbf{v}_i \,]^2}{\partial\, \mathbf{r}_i} \;=\; -\, m_i\, \mathbf{a}_i$$

where $\mathbf{r}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ are the inertial position, the inertial velocity and the inertial acceleration of particle i.

* In the temporal partial derivative, the spatial coordinates must be treated as constants [or replace this in the first equation: $+\,{}^1\!/_2\ \vec{v}_i \times \mathbf{B}$, and this in the second equation: $+\,{}^1\!/_2\ \vec{v}_i \times (\nabla \times \mathbf{A})$] $(\partial \mathbf{v}_s / \partial t \rightarrow \mathbf{a}_s)$ [or replace in the first equation: $+\,{}^1\!/_2\ (\vec{v}_i + \mathbf{v}_s) \times \mathbf{B}$, and in the second equation: $+\,{}^1\!/_2\ (\vec{v}_i + \mathbf{v}_s) \times (\nabla \times \mathbf{A})$]

Appendix C

Fields and Potentials III

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j can also be expressed as follows:

$$\mathbf{K}^a_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[\, \mathbf{E} + (\vec{v}_i - \vec{v}_j) \times \mathbf{B} \,\Big]$$

$$\mathbf{K}^a_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{v}_i - \vec{v}_j) \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}^a_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[-(\vec{a}_i - \vec{a}_j) + 2\,\vec{\omega} \times (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,] + \vec{\alpha} \times (\vec{r}_i - \vec{r}_j) \Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^1\!/\!_2\, [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,]^2 \;+\; {}^1\!/\!_2\, (\vec{v}_i - \vec{v}_j)^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,] + (\vec{v}_i - \vec{v}_j)$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; -\,\vec{\alpha} \times (\vec{r}_i - \vec{r}_j) + (\vec{a}_i - \vec{a}_j)\ ^*$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j can also be obtained starting from the following kinetic energy:

$$K^a_{ij} \;=\; -\, m_i\, m_j\, M^{-1} \big[\, \phi - (\vec{v}_i - \vec{v}_j) \cdot \mathbf{A} \,\big]$$

$$K^a_{ij} \;=\; {}^1\!/\!_2\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j) \,\big]^2$$

$$K^a_{ij} \;=\; {}^1\!/\!_2\, m_i\, m_j\, M^{-1} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2$$

Since the kinetic energy K^a_{ij} must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}^a_{ij} \;=\; -\,\frac{d}{dt} \left[\frac{\partial\, {}^1\!/\!_2\, \frac{m_i\, m_j}{M} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2}{\partial\, [\, \mathbf{v}_i - \mathbf{v}_j \,]} \right] + \frac{\partial\, {}^1\!/\!_2\, \frac{m_i\, m_j}{M} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2}{\partial\, [\, \mathbf{r}_i - \mathbf{r}_j \,]} \;=\; -\,\frac{m_i\, m_j}{M} \big[\, \mathbf{a}_i - \mathbf{a}_j \,\big]$$

where $\mathbf{r}_i, \mathbf{v}_i, \mathbf{a}_i, \mathbf{r}_j, \mathbf{v}_j$ and $\mathbf{a}_j$ are the inertial positions, the inertial velocities and the inertial accelerations of particles i and j.

* In the temporal partial derivative, the spatial coordinates must be treated as constants [or replace this in the first equation: $+\,{}^1\!/\!_2\, (\vec{v}_i - \vec{v}_j) \times \mathbf{B}$, and this in the second equation: $+\,{}^1\!/\!_2\, (\vec{v}_i - \vec{v}_j) \times (\nabla \times \mathbf{A})$]

Appendix D

Fields and Potentials IV

The kinetic force $\mathbf{K}_i^u$ exerted on a particle i of mass m_i by the center of mass of the Universe can also be expressed as follows:

$$\mathbf{K}_i^u \;=\; +\, m_i \Big[\, \mathbf{E} + (\vec{V}_{cm} - \vec{V}) \times \mathbf{B} \,\Big]$$

$$\mathbf{K}_i^u \;=\; +\, m_i \Big[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{V}_{cm} - \vec{V}) \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_i^u \;=\; +\, m_i \Big[-(\vec{A}_{cm} - \vec{A}) + 2\,\vec{\omega} \times (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{R}_{cm} - \vec{R})\,] + \vec{\alpha} \times (\vec{R}_{cm} - \vec{R}) \Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^1\!/_2\,[\,\vec{\omega} \times (\vec{R}_{cm} - \vec{R})\,]^2 \;+\; {}^1\!/_2\,(\vec{V}_{cm} - \vec{V})^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{R}_{cm} - \vec{R})\,] \;+\; (\vec{V}_{cm} - \vec{V})$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; -\,\vec{\alpha} \times (\vec{R}_{cm} - \vec{R}) + (\vec{A}_{cm} - \vec{A}) \;*$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{R}_{cm} - \vec{R})\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

The kinetic force $\mathbf{K}_i^u$ exerted on a particle i of mass m_i by the center of mass of the Universe can also be obtained starting from the following kinetic energy:

$$K_i^u \;=\; -\, m_i \,[\, \phi - (\vec{V}_{cm} - \vec{V}) \cdot \mathbf{A} \,]$$

$$K_i^u \;=\; {}^1\!/_2\, m_i \,\big[\,(\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})\,\big]^2$$

$$K_i^u \;=\; {}^1\!/_2\, m_i \,\big[\,\mathbf{V}_{cm}\,\big]^2$$

Since the kinetic energy K_i^u must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i^u \;=\; -\,\frac{d}{dt}\left[\frac{\partial\, {}^1\!/_2\, m_i\,\big[\,\mathbf{V}_{cm}\,\big]^2}{\partial\,\mathbf{V}_{cm}}\right] + \frac{\partial\, {}^1\!/_2\, m_i\,\big[\,\mathbf{V}_{cm}\,\big]^2}{\partial\,\mathbf{R}_{cm}} \;=\; -\, m_i\,\mathbf{A}_{cm}$$

where $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$ and $\mathbf{A}_{cm}$ are the inertial position, the inertial velocity and the inertial acceleration of the center of mass of the Universe.

* In the temporal partial derivative, the spatial coordinates must be treated as constants [or replace this in the first equation: $+\,{}^1\!/_2\,(\vec{V}_{cm} - \vec{V}) \times \mathbf{B}$, and this in the second equation: $+\,{}^1\!/_2\,(\vec{V}_{cm} - \vec{V}) \times (\nabla \times \mathbf{A})$]

Diagram I

Paper III		Paper IV
↓		↓
Newton's Law III (False)	=	Newton's Law III (False)
↓		↓
Auxiliary system of particles Free-System	=	Auxiliary system of particles Free-System
↓		↓
Invariant Magnitudes Inertial position : $\mathbf{r}_i$ Inertial velocity : $\mathbf{v}_i$ Inertial acceleration : $\mathbf{a}_i$	=	Invariant Magnitudes Inertial position : $\mathbf{r}_i$ Inertial velocity : $\mathbf{v}_i$ Inertial acceleration : $\mathbf{a}_i$
		↓
↓		Kinetic force $\mathbf{K}^a$ $\mathbf{K}_{ij}^a = -\,\dfrac{m_i\,m_j}{M}\,(\mathbf{a}_i - \mathbf{a}_j)$ Kinetic force $\mathbf{K}^u$ $\mathbf{K}_i^u = -\,m_i\,\mathbf{A}_{cm}$ Net kinetic force $\mathbf{K}_i = -\,m_i\,\mathbf{a}_i$
		↓
Second Principle $\mathbf{F}_i = m_i\,\mathbf{a}_i$		Second Principle $\mathbf{T}_i = \mathbf{K}_i + \mathbf{F}_i = 0$
↓		↓
Base Equation $\mathbf{F}_i = m_i\,\mathbf{a}_i$	=	Base Equation $\mathbf{F}_i = m_i\,\mathbf{a}_i$

Diagram II

$$\begin{tikzcd}
\text{Paper III} \arrow[d] & \text{Paper IV} \arrow[d] \\
\text{Base Equation} \arrow[d] \arrow[r, phantom, "="] & \text{Base Equation} \arrow[d] \\
\text{Equation of Motion} \arrow[d] \arrow[r, phantom, "="] & \text{Equation of Motion} \arrow[d] \\
\text{Definitions} \arrow[d] \arrow[r, phantom, "="] & \text{Definitions} \arrow[d] \\
\text{Relations} \arrow[d] \arrow[r, phantom, "="] & \text{Relations} \arrow[d] \\
\text{Conservation Laws} \arrow[d] \arrow[r, phantom, "="] & \text{Conservation Laws} \arrow[d] \\
\text{General Observations} \arrow[d] \arrow[r, phantom, "="] & \text{General Observations} \arrow[d] \\
\text{Annexes} \arrow[r, phantom, "="] & \text{Annexes} \arrow[d] \\
\begin{array}{ll} \nabla \cdot \mathbf{E} = 2\,\vec{\omega}^{\,2} & , \ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t & , \ \nabla \times \mathbf{B} = 0 \end{array} & \text{Appendices} \arrow[l]
\end{tikzcd}$$