Transformation of a Singular "Event Horizon" to any Radius Value: Reductio Ad Absurdum of Transformation Covariance Being More than a Dynamic Symmetry

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Abstract Due to David Hilbert's 1918 promotional efforts, J. Droste's May 27, 1916 metric solution of the Einstein equation for a static point mass fixed to the origin is universally featured by gravity textbooks, but is seriously misrepresented by those textbooks as the work of Karl Schwarzschild: Droste's metric solution has a well-known singular "event horizon" at the Schwarzschild radius, but Schwarzschild's January 13, 1916 metric solution is singular only at the origin. Here we present a simple family of transformations of Droste's singular "event horizon" to any radius value whatsoever. The blatant absurdity of, say, the earth's gravitational field having a singular "event horizon" at some arbitrary height above the earth's surface establishes beyond all doubt that gravitational general coordinate transformation covariance is, exactly like electromagnetic gauge transformation covariance, merely a dynamic symmetry which is unavoidably broken by the unique physical metric solution.

1. Transformation of the Droste metric's singular "event horizon" to an arbitrary radius value

On May 27, 1916, J. Droste published the following metric solution for a static point mass M fixed to $R = 0$ [1],

$$(c\,d\tau)^2 = (1 - r_s/R)(c\,dt)^2 - (1/(1 - r_s/R))(dR)^2 - R^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \tag{1.1a}$$

where $r_s \stackrel{\text{def}}{=} 2GM/c^2$ is called the Schwarzschild radius. The famous mathematician David Hilbert strongly promoted the Eq. (1.1a) metric in 1918 because of its algebraic simplicity, so gravity textbooks universally feature Droste's Eq. (1.1a) metric, but they seriously misrepresent it as the work of Karl Schwarzschild [2].

In Droste's Eq. (1.1a) metric, $g_{00}(R) = (1 - r_s/R)$, which vanishes at $R = r_s$, *so clocks at the Schwarzschild radius r_s are apparently stopped, the Droste metric's well-known "event horizon" phenomenon.* Furthermore, in Droste's metric, $g_{RR}(R) = (1/(1 - r_s/R))$, *which has a singularity at $R = r_s$, that same Schwarzschild radius.* Most *extended objects* such as the earth, sun and familiar stars *can't have such a singular "event horizon" since they lack an external Schwarzschild radius,* but we shall now exhibit a radius transformation $R(r)$ *which sends the Droste metric's singular "event horizon" at the Schwarzschild radius r_s to an arbitrary radius $r_a \geq 0$. A key property of this transformation $R(r)$ is that $R = r_s$ corresponds to $r = r_a$, so $R(r = r_a) = r_s$.* The Eq. (1.1a) Droste metric has determinant -1, *a property which the transformation $R(r)$ preserves.* Inserting $R(r)$ into Droste's Eq. (1.1a) metric produces,

$$(c\,d\tau)^2 = (1 - r_s/R(r))(c\,dt)^2 - (1/(1 - r_s/R(r)))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \tag{1.1b}$$

which is equivalent to,

$$(c\,d\tau)^2 = (1 - r_s/R(r))(c\,dt)^2 - (1/(1 - r_s/R(r)))(dR(r)/dr)^2(dr)^2 - (R(r)/r)^2 r^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big). \tag{1.1c}$$

The determinant of the transformed metric is $-1 \iff (dR(r)/dr)^2(R(r)/r)^4 = 1 \iff R^2 dR = \pm r^2 dr$. On selecting $\pm = +$, we obtain $(R(r))^3 = r^3 + (r_0)^3$, where r_0 is an arbitrary constant, so $R(r) = (r^3 + (r_0)^3)^{\frac{1}{3}}$. Since $R(r = r_a) = r_s$, $(r_a^3 + (r_0)^3)^{\frac{1}{3}} = r_s \implies (r_0)^3 = r_s^3 - r_a^3 \implies R(r) = (r^3 - r_a^3 + r_s^3)^{\frac{1}{3}} \implies dR(r)/dr = r^2(r^3 - r_a^3 + r_s^3)^{-\frac{2}{3}} = (r/(r^3 - r_a^3 + r_s^3)^{\frac{1}{3}})^2$. Putting these results for $R(r)$ and $dR(r)/dr$ into Eq. (1.1c) yields,

$$(c\,d\tau)^2 = \big(1 - r_s/(r^3 - r_a^3 + r_s^3)^{\frac{1}{3}}\big)(c\,dt)^2 -$$

$$\big(1/\big(1 - r_s/(r^3 - r_a^3 + r_s^3)^{\frac{1}{3}}\big)\big)(r/(r^3 - r_a^3 + r_s^3)^{\frac{1}{3}})^4(dr)^2 - ((r^3 - r_a^3 + r_s^3)^{\frac{1}{3}}/r)^2 r^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big). \tag{1.1d}$$

In the Eq. (1.1d) transformed metric, the Eq. (1.1a) Droste metric's singular "event horizon" at $R = r_s$ *has moved to $r = r_a$, where $r_a \geq 0$ is completely arbitrary,* entirely unlike $r_s = 2GM/c^2$.

When $r_a > r_s$, the Eq. (1.1d) transformed metric *has another singularity at* $r = (r_a^3 - r_s^3)^{\frac{1}{3}}$. The Eq. (1.1d) transformed metric *also always has a singularity at* $r = 0$. When $r_a = r_s$, the Eq. (1.1d) transformed metric *of course is merely the* Eq. (1.1a) *Droste metric itself.* When $r_a = 0$, the Eq. (1.1d) transformed metric *is singular only at* $r = 0$; in fact when $r_a = 0$ the Eq. (1.1d) transformed metric *is precisely the metric solution published by Karl Schwarzschild in his January 13, 1916 paper*—a 1999 English translation of Schwarzschild's paper is available as arXiv:physics/9905030v1 [physics.hist-ph] 12 May 1999 [3].

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Eq. (1.1d) *is an r_a-parameterized infinite family of metric solutions of the Einstein equation for a static point mass M fixed to $r = 0$.* In conjunction with the Birkhoff theorem, Eq. (1.1d) *absurdly suggests that the earth can have a singular "event horizon" at any height whatsoever above its surface!* But *since the physical metric solution for a static point mass M fixed to $r = 0$ is unavoidably unique,* gravitational general coordinate transformation covariance *necessarily is merely a dynamic symmetry*[4] *which is unavoidably broken by the unique physical metric solution.*

To zero in on what it is that selects the unique physical metric solution for a static point mass M fixed to $r = 0$, *we temporarily detour to the Newtonian-gravity potential solution* $\Phi(r)$ for a static point mass M fixed to $r = 0$. This Newtonian-gravity potential solution $\Phi(r)$ satisfies the equation,

$$\nabla^2_{\mathbf{x}}\Phi(|\mathbf{x}|) = 4\pi G\,M\delta^{(3)}(\mathbf{x}), \tag{1.2a}$$

which is the Newtonian-gravity *analog of the Einstein equation* for a static point mass M fixed to $r = 0$. When $|\mathbf{x}| > 0$, Eq. (1.2a) is simply,

$$\nabla^2_{\mathbf{x}}\Phi(|\mathbf{x}|) = 0, \tag{1.2b}$$

which implies that when $r > 0$, $\Phi(r)$ is twice differentiable, so when $r > 0$, $\Phi(r)$ is singularity-free. This inference is fully borne out by the well-known exact solution of Eq. (1.2a),

$$\Phi(r) = -GM/r. \tag{1.2c}$$

With regard now to the Einstein equation for a static point mass M fixed to $r = 0$, it is obvious that when $r > 0$, the Einstein tensor, the curvature scalar and the Ricci curvature tensor all vanish. Therefore when $r > 0$, the components of the metric tensor must all be twice differentiable and thus singularity-free. Karl Schwarzschild's January 13, 1916 metric solution for a static point mass M fixed to $r = 0$ *is the unique metric solution which both has determinant -1 and is singularity-free when $r > 0$.* As we have already mentioned, Karl Schwarzschild's January 13, 1916 metric solution for a static point mass M fixed to $r = 0$ *is the $r_a = 0$ member of the* Eq. (1.1d) *family of such metric solutions with determinant -1, namely,*

$$\begin{aligned}(c\,d\tau)^2 &= \big(1 - r_s/(r^3 + r_s^3)^{\frac{1}{3}}\big)(c\,dt)^2 - \\ &\big(1/\big(1 - r_s/(r^3 + r_s^3)^{\frac{1}{3}}\big)\big)(r/(r^3 + r_s^3)^{\frac{1}{3}})^4(dr)^2 - ((r^3 + r_s^3)^{\frac{1}{3}}/r)^2 r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big),\end{aligned} \tag{1.3a}$$

which is considerably more compactly written in its Eq. (1.1b) form,

$$(c\,d\tau)^2 = (1 - r_s/R(r))(c\,dt)^2 - (1/(1 - r_s/R(r)))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big), \tag{1.3b}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. The *problem* with making coordinate transformations of Schwarzschild's solution, such as those in the Eq. (1.1d) solution family, *is that they displace the metric's singular point at $r = 0$.*

We have stated that gravitational general coordinate transformation covariance *is merely a dynamic symmetry which is unavoidably broken by the unique physical metric solution.* It is therefore useful *to review how electromagnetic gauge transformation covariance dynamic symmetry is dealt with.*

2. Dealing with electromagnetic gauge transformation covariance dynamic symmetry

In four-vector potential electromagnetic theory, the analog of the Einstein equation is,

$$\partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\sigma A^\sigma) = 4\pi j^\mu/c. \tag{2.1a}$$

where the current density j^μ satisfies the equation of continuity $\partial_\mu j^\mu = 0$. As it is written, Eq. (2.1a) *isn't self-consistent unless $\partial_\mu j^\mu = 0$ because $\partial_\mu(\partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\sigma A^\sigma)) = \partial_\nu\partial^\nu(\partial_\mu A^\mu) - \partial_\mu\partial^\mu(\partial_\sigma A^\sigma) = 0$.* Furthermore, if A^μ is a solution of Eq. (2.1a), so is $A^\mu + \partial^\mu\chi$, where χ is an arbitrary scalar function, because $(\partial_\nu\partial^\nu\partial^\mu\chi - \partial^\mu(\partial_\sigma\partial^\sigma\chi)) = \partial^\mu(\partial_\nu\partial^\nu\chi - \partial_\sigma\partial^\sigma\chi) = 0$, *so* Eq. (2.1a) *suffers solution nonuniqueness.*

Among the solutions of Eq. (2.1a) *are ones which aren't Lorentz covariant.* For example, the Coulomb gauge condition $\nabla\cdot\mathbf{A}' = 0$ can always be satisfied by introducing the appropriete scalar gauge function χ, namely $\mathbf{A}' = \mathbf{A} + \nabla\chi$, where $\nabla^2\chi = -\nabla\cdot\mathbf{A}$. However, *since the Coulomb gauge condition $\nabla\cdot\mathbf{A}' = 0$ doesn't accord with Lorentz covariance, the solutions of* Eq. (2.1a) *which are consistent with the Coulomb gauge condition can't be expected to be Lorentz covariant.*

So *in order to ensure a solution of* Eq. (2.1a) *which is Lorentz covariant, we must impose a Lorentz-covariant gauge condition on that solution.* The simplest possible Lorentz-covariant gauge condition obviously is $\partial_\mu A'^\mu = 0$, which can always be satisfied by introducing the appropriate scalar gauge function χ, namely $A'^\mu = A^\mu + \partial^\mu \chi$, where $\partial_\mu \partial^\mu \chi = -\partial_\mu A^\mu$. *Imposing this Lorentz condition,* $\partial_\sigma A^\sigma = 0$, *on* Eq. (2.1a) *produces the following two equations,*

$$\partial_\sigma A^\sigma = 0, \tag{2.1b}$$

and,

$$\partial_\nu \partial^\nu A^\mu = 4\pi j^\mu / c. \tag{2.1c}$$

Eq, (2.1c) *doesn't have a mathematically unique solution because the differential operator* $\partial_\nu \partial^\nu$ *doesn't have a mathematically unique inverse, but only the "retarded" inverse* $(\partial_\nu \partial^\nu)_R^{-1}$ *of the operator* $\partial_\nu \partial^\nu$ *makes causal physical sense.* Thus *the unique physically sensible causal solution of* Eq. (2.1c) *is the "retarded" one*[5],

$$A^\mu = (4\pi/c)(\partial_\nu \partial^\nu)_R^{-1} j^\mu, \tag{2.1d}$$

which in addition satisfies Eq. (2.1b) *because* $\partial_\mu j^\mu = 0$.

Thus *solution nonuniqueness due to electromagnetic gauge transformation covariance dynamic symmetry is resolved by imposing absolutely basic physical requirements on the solution in the simplest feasible way.*

REFERENCES

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[3] arXiv:physics/9905030v1 [physics.hist-ph] 12 May 1999.

[4] S. Weinberg, *op cit,* p. 93.

[5] https://en.wikipedia.org/wiki/Retarded_potential.