Constructive QFT : a condensed math point of view III

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Abstract

In this series, we show a road map to prove the existence of ϕ^4 quantum field theory over 4-d spacetime. We suggest a new axiomatic approach on constructive quantum field theory via condensed mathematics. The goal is to extend this new approach to cover the mathematical viability of field theories of standard model and beyond.

III. Reminder of CQFT

1 Lattice regularization

Let's give a mathematically rigorous meaning to Schwinger functions,[1]

$$
\begin{aligned}
S_n(x_1, x_2, \cdots, x_n) &\equiv \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle \\
&= \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\cdots\phi(x_n)\ e^{-S[\phi]}}{\int \mathcal{D}\phi\ e^{-S[\phi]}}
\end{aligned}
$$

Note that $x \in X = \mathbb{R}^4$, and ϕ is a real scalar field over X. Feynman's path integral formalism starts from slicing time into infinitesimal. Each infinitesimal change in time gives the measure

$$\mathcal{D}\phi = \prod_x d\phi(x)$$

This measure can be legitimately defined over finite lattice Λ. Now the coordinates for Λ is $(\mathbb{Z}_1 a, \mathbb{Z}_2 a, \mathbb{Z}_3 a, \mathbb{Z}_4 a)$ with $\mathbb{Z}_\mu \ll \infty, \mu \in \{1, 2, 3, 4\}$. Let's abbreviate this as $z \in \Lambda = \mathbb{Z}_a^4$. A lattice action should be defined over this discrete spacetime. With

[1] We omit time ordering convention for a while. It is O.K. since our theory is only bosonic fields. We need the notation right when we generalize it to fermionic fields.

a given action, lattice version of correlators is established. Then the construction of continuum QFT is by taking the limit $a \to 0$.

Let us remind this process by showing how 2-point functions, or propagators, $\langle\phi(x_1)\phi(x_2)\rangle_{latt}$ is defined. General case can be achieved via Wick's theorem. The goal is to construct $\langle\phi(x_1)\phi(x_2)\rangle_{cont}$ at the end of this note. Our approach is to define a $\langle\phi(x_1)\phi(x_2)\rangle_{cond}$, which gives both $\langle\phi(x_1)\phi(x_2)\rangle_{latt}$ and $\langle\phi(x_1)\phi(x_2)\rangle_{cont}$ at its certain limit. So this note is for laying the background of the analytic definition of ϕ^4_{cond}-theory. Then we will change the gear to define it from the perspective of algebraic geometry.

In the last note, we introduced propagator of a free particle by Green's function solution, such that

$$G(x,y) = \frac{1}{(2\pi)^4}\int \frac{e^{ik(x-y)}}{k^2+m^2}\, d^4k$$

This solution is available with the measure defined on points of spacetime.[2] However, when the measure is over field operators, such as ϕ on spacetime, then the solution is not followed by the classical method. Instead, one requires to slice the spacetime into lattice then define the measure of field configurations rigorously.[3] Physically, lattice spacetime cuts down momentum space, so that the integration over field configuration space is not out of control.

We defined generating functional

$$Z[J] \equiv \int \mathcal{D}\phi\; e^{-(S[\phi]+\int dx\, J(x)\phi(x))}$$

with a free action by

$$S = \int d^4x\, \mathcal{L}_{free}, \quad \mathcal{L}_{free} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_B^2\phi^2$$

Then the propagator is

$$\langle\phi(x_1)\phi(x_2)\rangle_{cont} = \frac{1}{Z[0]}\left(\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right)_{J=0}$$

Let's call this propagator $\langle\phi(x_1)\phi(x_2)\rangle_{cont}$, for it is defined over continuum spacetime. As it was remarked shortly, in order to evaluate the integral of generating functional one requires to regularize the integral. So we change the starting point as follows.

$x \in X = \mathbb{R}^4 \to za \in \Lambda = \mathbb{Z}^4_a$, with z finite.

$\phi(x) \to \phi(za)$

$\partial_\mu \to \frac{\phi(za+\mu a)-\phi(za)}{a}$, with μ in Euclidean 4-directional vectors.

[2] Or equivalently, 4-momentum space.

[3] Regularization is a whole packages to define and solve the integral of generating functional. There is other methods such as dimensional regularization or Pauli-Villars regularization. For our purpose, we just treat lattice regularization as 'the' regularization.

$\int \mathcal{D}\phi(x) \to \int \prod_z d\phi(za)$

$S = \int d^4x\, \mathcal{L} \to S_{latt} = a^4 \sum_z \mathcal{L}_{latt}$

With this starting point, the generating functional is

$$Z[J]_{latt} \equiv \int \prod_z d\phi(za)\ e^{-(S[\phi]_{latt} + \sum_z\ J(za)\phi(za))}$$

And the propagator is

$$\langle \phi(z_1 a)\phi(z_2 a)\rangle_{latt} = \frac{\int \prod_z d\phi(za)\phi(z_1 a)\phi(z_2 a)e^{-S[\phi]_{latt}}}{\int \prod_z d\phi(za)e^{-S[\phi]_{latt}}}, \quad z_1, z_2 \in \mathbb{Z}^4_{fin}$$

2 Dimensional analysis

In physics, the numbers that one deals with has physical dimension. So that the numbers in mass, energy or length have different dimensions. For example, by path integral formalism

$$\langle x_f, t_f | x_0, t_0\rangle = \int_{x_0}^{x_f} d^3x\, e^{\frac{i}{\hbar}\int dtL}$$

Since the dimension of the Planck constant is equal to the dimension of the action, the exponential is dimensionless. So the integral has the dimension of length, which gives the physical meaning of expectation value of disposition.[4] By setting $\hbar = c = 1$, one also transforms the dimension of the given system. In this setting, one can only consider dimensions in a single parametrization. Let's use length based unit such that

$$[L] = [M^{-1}] = 1, \;\; [L^{-1}] = [M] = -1$$

Parameters and operators of equations have units such as

$$[x] = [dx] = 1, \;\; [p] = [m] = [\partial_x] = [\phi] = -1, \;\; [\lambda] = 0$$

With this setting, the exponetial is again dimensionless. And the ϕ^4 coupling λ becomes dimensionless. From this dimensionless parameter λ one can perturb the solution for ϕ^4 interacting theory.

The physical meaning of Schwinger function is the probability amplitudes of events. Observationally, what one calculates is dimensionless numbers. So Schwinger function should be dimensionless. But from the equation,

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\ e^{-S[\phi]}}{\int \mathcal{D}\phi\ e^{-S[\phi]}}$$

[4] Probability is dimensionless. And disposition is dimensional in length. For example, $[\langle x\rangle] = [\sum_x p(x)\cdot x] = [p(x)] + [x] = [x] = [L]$.

$$[\langle\phi(x_1)\phi(x_2)\rangle] = -2$$

This can be remedied by reparametrizing ϕ and other operators. It is done by making field operators and mass term to be dimensionless from the start. From the definition of free lattice action

$$S[\phi]_{latt} = \frac{a^4}{2}\sum_{z_1,z_2}\phi(z_1a)(\Box_{latt} - m_B^2)\phi(z_2a), \quad \Box = \partial_\mu\partial^\mu$$

The action is dimensionless. By distributing a to each terms, one gets

$$S[\phi]_{latt} = \frac{1}{2}\sum_{z_1,z_2}a\phi(z_1a)(a^2\Box_{latt} - a^2m_B^2)a\phi(z_2a)$$

Then one can redefine every components as

$$\hat{\phi}(za) = a\phi(za), \quad \hat{\partial} = a\partial, \quad \hat{\Box}_{latt} = a^2\Box_{latt}, \quad \hat{m}_B = am_B$$

$$[\hat{\phi}(za)] = [\hat{\partial}] = [\hat{\Box}_{latt}] = [\hat{m}_B] = 0$$

So the S_{latt} over this reparametrization becomes,

$$S[\hat{\phi}]_{latt} = \frac{1}{2}\sum_{z_1,z_2}\hat{\phi}(z_1a)(\hat{\Box}_{latt} - \hat{m}_B^2)\hat{\phi}(z_2a)$$

Now every components of generating functional is dimensionless so the dimensionless lattice propagator is

$$\langle\hat{\phi}(z_1a), \hat{\phi}(z_2a)\rangle_{latt} = \frac{\int\prod_z d\hat{\phi}(za)\hat{\phi}(z_1a)\hat{\phi}(z_2a)e^{-S[\hat{\phi}]_{latt}}}{\int\prod_z d\hat{\phi}(za)e^{-S[\hat{\phi}]_{latt}}}, \quad z_1, z_2 \in \mathbb{Z}^4_{fin}$$

$$[\langle\hat{\phi}(z_1a), \hat{\phi}(z_2a)\rangle_{latt}] = 0$$

Now, let's see how to construct $\langle\phi(x_1), \phi(x_2)\rangle_{cont}$ from $\langle\hat{\phi}(z_1a), \hat{\phi}(z_2a)\rangle_{latt}$.

3 Gaussian integral

Gaussian integral has n-dimensional general solution when the integral has the form,

$$\int d^n x\, e^{-\frac{1}{2}(x_iMx_j)+N_ix_i} = \sqrt{\frac{(2\pi)^n}{\det M}}\, e^{\frac{1}{2}N^\dagger M^{-1}N}$$

From lattice action S_{latt}, the kernel $K = (\hat{\Box}_{latt} - \hat{m}_B^2)$ can be rewritten as the following steps,

$$\hat{\partial}_\mu\hat{\phi}(za) = \sum_\mu \hat{\phi}(za + \mu a) - \hat{\phi}(za)$$

$$\hat{\partial}^\mu \hat{\partial}_\mu \hat{\phi}(za) = \sum_\mu \hat{\phi}(za + 2\mu a) - \hat{\phi}(za + \mu a) - \hat{\phi}(za + \mu a) + \hat{\phi}(za)$$

Since $S[\hat{\phi}]$ is the sum over all z, one can reparametrize it to get

$$\sum_z \hat{\partial}^\mu \hat{\partial}_\mu \hat{\phi}(za) = \sum_{z,\mu} \hat{\phi}(za + \mu a) + 2\hat{\phi}(za) - \hat{\phi}(za - \mu a)$$

With mass term combined, lattice action becomes

$$S[\hat{\phi}]_{latt} = \sum_{z,\mu} \hat{\phi}(za) \left(\hat{\phi}(za + \mu a) + 2\hat{\phi}(za) - \hat{\phi}(za - \mu a) \right) - \hat{m}_B^2 \hat{\phi}(za)$$

One can rewrite this in a matrix kernel K_{z_1,z_2} as

$$K_{z_1,z_2} = -\sum_\mu (\delta_{z_1,z_2+\mu a} + \delta_{z_1,z_2-\mu a} - 2\delta_{z_1,z_2}) + \hat{m}_B^2 \delta_{z_1,z_2}$$

By writing action with this matrix kernel,

$$S_{latt} = \frac{1}{2} \sum_{z_1,z_2} \hat{\phi}(z_1 a) K_{z_1,z_2} \hat{\phi}(z_2 a)$$

One has the solution of latticed generating functional,

$$\begin{aligned} Z[J]_{latt} &= \int \prod_z d\hat{\phi}(za)\, e^{-(\frac{1}{2}\sum_{z_1,z_2} \hat{\phi}(z_1 a) K_{z_1,z_2} \hat{\phi}(z_2 a) + \sum_z J(za)\hat{\phi}(za))} \\ &= \sqrt{\frac{1}{\det K}}\, e^{\frac{1}{2}\sum_{z_1,z_2} J(z_1) K^{-1}_{z_1,z_2} J(z_2)} \end{aligned}$$

With this solution, the lattice propagator is

$$\begin{aligned} \langle \hat{\phi}(z_1)\hat{\phi}(z_2) \rangle_{latt} &= \frac{1}{Z[0]} \left(\frac{\delta^2 Z[J]}{\delta J(z_1) \delta J(z_2)} \right)_{J=0} \\ &= K^{-1}_{z_1,z_2} \end{aligned}$$

Now, in order to get $K^{-1}_{z_1,z_2}$,

$$\sum_z K_{z_1,z} K^{-1}_{z,z_2} = \delta_{z_1,z_2}$$

δ_{z_1,z_2} is via Fourier transformation to dimensionless momentum space $\hat{k}_\mu = a k_\mu$,[5]

$$\delta_{z_1,z_2} = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4}\, e^{i\hat{k}\cdot(z_1 - z_2)}$$

[5]Note that the range of integration is restricted by $[-\pi, \pi]$. Because the position is latticed, Fourier transform is as follows, $f(na) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \tilde{f}_a(k)\, e^{ikna}$.

Then applying this to the K_{z_1,z_2} to get

$$K_{z_1,z_2} = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} \left(4 \sum_{\mu} \sin \frac{\hat{k}_\mu}{2} + \hat{m}_B^2 \right) e^{i\hat{k}\cdot(z_1 - z_2)}$$

In the end, the solution of lattice propagator becomes

$$\langle \hat{\phi}(z_1 a) \hat{\phi}(z_2 a) \rangle_{latt} = K^{-1}_{z_1,z_2} = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} \frac{e^{i\hat{k}\cdot(z_1-z_2)}}{4\sum_\mu \sin \frac{\hat{k}_\mu}{2} + \hat{m}_B^2}$$

4 Propagator construction from lattice to continuum

Now let's check how lattice propagator transform after continuum limit is taken. To remove spacetime lattice one changes back to the limit such as

$$\hat{\phi} \to a\phi,\ za \to x,\ \hat{k} \to ak,\ \hat{m}_B \to am_B,$$

lattice propagator approximates to

$$\langle \hat{\phi}(z_1 a) \hat{\phi}(z_2 a) \rangle_{latt} \approx a^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 + M^2}$$

It becomes continuum propagator at $a \to 0$ limit

$$\langle \hat{\phi}(z_1 a) \hat{\phi}(z_2 a) \rangle_{latt} \approx a^2 \langle \phi(x_1) \phi(x_2) \rangle_{cont}$$

So that the continuum propagator is constructed via lattice propagator such that

$$\langle \phi(x_1) \phi(x_2) \rangle_{cont} = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2}$$

Note that removing the gap of lattice makes the range of integral to $[-\infty, \infty]$, and the dependency of a is not there in this solution.

From the last note, the propagator of free particle is Green's function solution such that,

$$(\Box + m^2) G(x, y) = \delta(x - y)$$

the propagator of continuum spacetime in momentum space is

$$\langle \phi(x_1) \phi(x_2) \rangle_{cont} = G(x, y) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2}$$

which is the same result from lattice construction. So in this simple scalar field propagators, continuum QFT is constructible from the lattice QFT.[6]

[6]The observable is actually dimensionless number such as the ratio $\langle \phi(x_1) \phi(x_2) \rangle_{cont} / \mu^2$. Lattice version is $\langle \hat{\phi}(z_1) \hat{\phi}(z_2) \rangle_{cont} / \hat{\mu}^2$, and it is independent on $\hat{\mu}$ if it is sufficienty small.

5 Construction of ϕ^4 interacting theory

So far the theory has no interaction terms, so that the probability amplitudes can be summed by propagators only. And the existence is guaranteed by the existence of propagators. Fields or particles in the nature interact each other, so the Lagrangian is required to contain such interacting terms. We study the Lagrangian with ϕ^4 interacting term, and call it just simply ϕ^4-theory from now on.

Our Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_B^2\phi^2 + \frac{1}{4!}\lambda_B\phi^4$$

B is for bare. We want to emphasize that two constants can be variant up to scale factor. But let's suppose coupling constant λ is fixed for now, with $\lambda \ll 0$. Then perturbative expansion method can be used to approximate the solution of ϕ^4-theory.[7]

For example propagator of ϕ^4-theory is

$$\langle\phi(x_1)\phi(x_2)\rangle_{cont} = \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\ e^{-S[\phi]}}{\int \mathcal{D}\phi\ e^{-S[\phi]}}$$

We now separate action as

$$S[\phi] = S[\phi]_{free} + S[\phi]_{int}$$

Then taking power series over $e^{-S[\phi]_{int}} = e^{-\int d^4x \frac{1}{4!}\lambda\phi(x)^4}$,

$$\langle\phi(x_1)\phi(x_2)\rangle_{cont} = \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\ e^{-S[\phi]_{free}}(1 - \int d^4x \frac{\lambda}{4!}\phi(x)^4 + \cdots)}{\int \mathcal{D}\phi\ e^{-S[\phi]_{free}}(1 - \int d^4x \frac{\lambda}{4!}\phi(x)^4 + \cdots)}$$

For physical reason, the operators should be in time order.[8] And the creation and annihilation operators of field component also should be in right order to make the contribution not zero such as $\langle 0|a_1a_2\cdots a_1^\dagger a_2^\dagger|0\rangle$. By Wick's theorem, timely ordered events with non-zero contributions are the same as the contributions out of contractions. Contractions are the combinations of propagators that support the Feynman's diagram chasing method.

For example,

$$\langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle_{cont,free} = Graph_1 + Graph_2 + Graph_3$$

It has 3 contributions from the figure. And each contribution is the combinations of propagators, so that

$$\langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle_{cont,free} = G_0(x,z)G_0(y,x) + G_0(x,y)G_0(z,w) + G_0(x,w)G_0(y,z)$$

[7] For mathematical existence, there is divergence issues in perturbative integrals. We remind renormalization in the next section.

[8] It is by proper time. In our notation time ordering factor is omitted. Every operators inside the braket should be considered as time ordered operators acting on the vacuum. $\langle\phi_1\phi_2\rangle = \langle\Omega|\mathcal{T}(\phi_1\phi_2)|\Omega\rangle$.

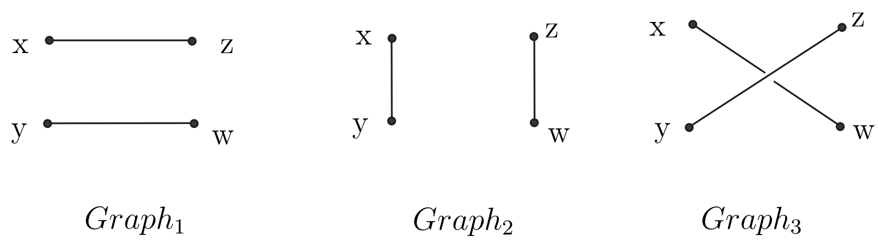


Fig. 1

Subscript 0 is for propagators of free action. So let's use the same notation for correlators.

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle_{cont,free} = \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle_0$$

Now back to the the propagator of ϕ^4-theory is

$$\langle \phi(x_1)\phi(x_2)\rangle_{cont} =$$

$$\frac{\langle \phi(x_1)\phi(x_2)\rangle_0 - \frac{\lambda}{4!}\int d^4x\,\langle \phi(x_1)\phi(x_2)\phi^4(x)\rangle_0 + \frac{1}{2!}(\frac{\lambda}{4!})^2\int d^4xd^4x'\,\langle \phi(x_1)\phi(x_2)\phi^4(x)\phi^4(x')\rangle_0 - \cdots}{1 - \frac{\lambda}{4!}\int d^4x\,\langle \phi^4(x)\rangle_0 + \frac{1}{2!}(\frac{\lambda}{4!})^2\int d^4xd^4x'\,\langle \phi^4(x)\phi^4(x')\rangle_0 - \cdots}$$

Let's show some components of the fraction, Via Wick's theorem,

$$\langle \phi(x_1)\phi(x_2)\rangle_0 = G_0(x_1, x_2) \tag{1}$$

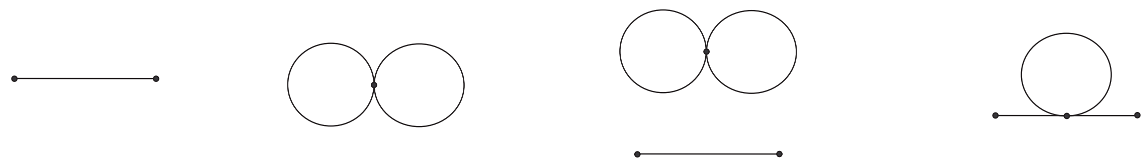
$$\langle \phi(x)\phi(x)\phi(x)\phi(x)\rangle_0 = 3\int d^4x\, G_0(x,x)G_0(x,x) \tag{2}$$

$$\langle \phi(x_1)\phi(x_2)\phi(x)\phi(x)\phi(x)\phi(x)\rangle_0 = 3\int d^4x\, G_0^2(x,x)G_0(x_1,x_2) \tag{3}$$

$$+\,12\int d^4x\, G_0(x_1,x)G_0(x,x)G_0(x,x_2) \tag{4}$$

Note that coefficient in front of integral is about counting the number of possible contractions. 3 for (2) is by choosing a $\phi(x)$, there is 3 possible choices to pick another operator. Once two out of 4 fields are chosen, then last contraction is determined. For 6 particles, one creation can find 5 ways to annihilate. Then 4 particles remain 3 ways to make, etc. So total 15 ways of contractions is possible. (3) has 4 fields of degree of choices and a propagator, so the coefficient is 3. (4) has x_1 to choose 4 out of x, then 3 remaining x to choose x_2, so the coefficient is 12. This degree of choices gets bigger as the perturbation is higher in degrees. So the perturbation itself cannot converge for any λ, as it is. This should be resolved by the renormalization.

Note that the denominator contains those terms without external lines. These are all composed of bubbles. On the other hand, the numerator contains connected diagrams and also disconnected diagrams with bubbles. In the end, the denominator cancels out



(1) (2) (3) (4)

Fig. 2: Each diagram represents the integrals (1),(2),(3),(4).

all the disconnected part of numerators. So the calculation becomes the sum of all the connected graphs. Feynman rules are the algorithm to trace the calculations up to the degrees of λ^n terms. For calculational reason, momentum space perturbation is usually used. And the Feynman rules for the momentum space diagrams is such that

- Input and output momentum vectors for each external lines.
- propagators for each internal lines.
- interacting contribution at a four point vertex with λ factor.
- the sum of the momentum at each vertex is conserved.

It is only simplified statements. One can add more details for calculational purpose. But for our constructive purpose, we need to focus on the existence of propagators in internal lines. These internal propagators can plague the theory very easily. For example, for the graph of the Fig.2 (2), one writes the contribution as

$$-3\frac{\lambda}{4!}\int d^4x\, G_0(x,x)G_0(x,x) = -3\frac{\lambda}{4!}\int d^4x \int \left(\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\right)^2$$

If d^4x is over $\mathbb{R}^4$, then the integral is divergence. So d^4x should be over finite volume space Λ. Even with finite volume space, the total sum of contributions can be divergent with big enough volume factor. This is IR divergence problem. Also the propagator inside diverges with large k. It is required to cut off large momentum. This is UV divergence problem. In order to evade these two divergent issue, one construct QFT from a regularized spacetime. A lattice with finite volume can be one of it.

Regularizing spacetime is not enough to excise every infinite contributions of interacting theory. There is another divergence factor at the higher degree of perturbations by the permutations of choices on fields. One requires to excise every divergent contributions by renormalizing fields, mass and coupling constant. One has to deal with three types of divergences in interacting correlators. The existence of interacting theory can be decided whether one can remove these pathologies. If it is possible then the theory is called renormalizable.

6 Renormalization

To give the correlators of ϕ^4-theory legitimate meaning, renormalization is necessary to clear the infinities out of perturbation theory. The contributions of perturbative theory are graded by degrees of interactions. As noted, summing entire contributions of every possible events gives many inifinite integrals. So one should excise singular terms to get finite contributions of each degree of Feynman graphs. This can be done by adding counter terms to the actions for deleting divergent terms.

$$S_{renormalized} = S_{bare} - S_{counter}$$

To motivate the process, let's approach it from rescaling field parameters first,

$$\phi = \sqrt{Z}\tilde{\phi}$$

By replacing field operator with renormalized field operator $\sqrt{Z}\tilde{\phi}$, the Lagrangian density becomes

$$\mathcal{L}_R = \frac{Z}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{Z}{2}m_B^2\tilde{\phi}^2 + \frac{Z^2}{4!}\lambda_B\tilde{\phi}^4$$

Note that, renormalizing factor Z can be defined by functions of lattice spacing a. So the theory depends on the construction of lattice to continuum limit by letting $a \to 0$. Now by introducing renormalized parameters m, λ, the renormalized Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_R = &\left(\frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{1}{2}m^2\tilde{\phi}^2\right)_{free} \\ &+ \left(\frac{1}{4!}\lambda\tilde{\phi}^4 + (Z-1)\frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - (m_B^2 - m^2)\frac{1}{2}\tilde{\phi}^2 - (\lambda_B - \lambda)\frac{1}{4!}\tilde{\phi}^4\right)_{int}\end{aligned}$$

By this reformulation of Lagrangian density, one can trace the counter terms inside interaction part. Perturbation can be performed over renormalized coupling constant. The theory which can be constructed by renormalization procedure is called renormalizable. Renormalizability is the first step to construct the quantum field theory with interactions. With extra axioms satisfied by the field theory, which is about making the field theory relativistic and justified with observations, one can have the QFT existence.

In the next note, we will start to define *condensed* quantum field theory.

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