An Elementary Solution of the Navier-Stokes Existence and Smoothness

Masatoshi Ohrui (大類昌俊)

(MasatoshiOhrui1993@gmail.com)

Abstract

This is an elementary argment in the sense that there are no long or complicated calculations, and the theory of evolution equations is not used at all. Our initial values can be taken arbitrary large, our solutions are time global and physically suitable.

Introduction

The existence of the solutions is actually known. For example, Fujita-Kato Theory, Shibata Theory: Zhang [30], Charve-Danchin [31], Shibata-Miura [32]. The semi-group theory or apriori estimates are in these theories, but these are not elementary. The initial values can be taken arbitrary large in the Leray-Hopf's weak solutions, but the uniqueness and smoothness are unresolved. Semi-group theory or apriori estimates are not used in the proof of existence of the Leray-Hopf's weak solutions (for example, Wasao SIBAGAKI, Hisako RIKIMARU [29]), but it is not elementary, too. We define new weak solutions with uniqueness and smoothness, without semi-group theory or apriori estimates. The initial values mast be small in the Fujita-Kato theory or Shibata theory to show the solutions are time global, but our initial values can be taken arbitrary large and our solutions are time global. We apply locally solvability of the partial differential oparators with constant coefficients. The policy is, to let L be the heat operator $\partial_t - \Delta$ in the initial value problem of the Navier-Stokes equations on Ω

$$\partial_t u - \Delta u = f - \nabla \mathfrak{p} - (u \cdot \nabla) u$$
$$\operatorname{div} u = 0$$
$$u(0, x) = a(x) \ ,$$

to erase the pressure $\mathfrak{p}$, to approximate the nonlinear term $(u \cdot \nabla) u$ by a sequence of smooth functions, to use the locally solvability for the difference between the external force f and the approximation term, and to show that the limit in the Sobolev space is the solution. Our solutions are physically suitable: $f \mapsto u, f \mapsto \mathfrak{p}; a \mapsto u, a \mapsto \mathfrak{p}$ are continuous, and $\lim_{t,|x| \to \infty} \partial^\alpha u(t,x) = 0$.

[Definition of symbols]
For convenience, we write the index of the component of the vectors in the upper right corner. "Function space" and "space" are abbreviations for "linear topological space" (of functions or distributions), other than pressure $\mathfrak{p}$ are $\mathbb{R}^3$ -valued. The absolute value of the functions in the norm of normal function space is interpreted as the length of the number vector (the absolute value of $\mathbb{R}^3$) in the norm of the space of the $\mathbb{R}^3$ -value functions. We write the space of the real numeric functions and the space of the $\mathbb{R}^3$ -value functions in the same symbol to make symbols simple. For any positive number δ , let $B_\delta(0,0)$ be the δ -neighborhood of point $(0,0)$. Let $|\Omega|$ be Ω 's Lebesgue measure. Let χ_Ω be the characteristic function on Ω . For any natural number $m > \max\{0 + 4/1, 0 + 4/2\} = 4$, $p = 1, 2$, let $V^{m,p}_\sigma(\Omega) = \{u \in C^\infty(\mathbb{R} \times \mathbb{R}^3) : \|u\|_{W^{m,p}(\Omega)} < \infty, \operatorname{div} u = 0\}$, $W^{m,p}_\sigma(\Omega)$ be the Sobolev space defined by $V^{m,p}_\sigma(\Omega)$'s completion by norm of $W^{m,p}_\sigma(\Omega) = \overline{V^{m,p}_\sigma(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}$. Let $\mathcal{D}(\Omega)$ be the space of the test functions ($C_0^\infty(\Omega)$ as a set), let $\mathcal{D}_\sigma(\Omega)$ be the space of the test functions that the divergence is 0 for spatial variables (see [Supplement 1]). Let $P : L^2(\Omega) \to L^2_\sigma(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{\|\cdot\|_{L^2(\Omega)}}$ be the projection. Let $C^{k,\varepsilon}(\overline{\Omega})$ be the Hölder space. Let
$\langle w, \varphi \rangle = (w, \varphi)_{L^2(\Omega)}$
$= \int_\Omega \sum_{i=1}^3 w^i(t,x) \varphi^i(t,x) dt dx$
$= \int_\Omega w(t,x) \cdot \varphi(t,x) dt dx$
$(w = (w^1, w^2, w^3), \varphi = (\varphi^1, \varphi^2, \varphi^3))$.
In general, if for two Banach spaces X, Y , there exists a linear Hausdorff space Z such that $X, Y \subset Z$, then $X \cap Y$ is a Banach space with norms given by $\|u\|_X + \|u\|_Y$ or $\max\{\|u\|_X, \|u\|_Y\}$. $\max\{\|u\|_X, \|u\|_Y\} \le \|u\|_X + \|u\|_Y \le 2\max\{\|u\|_X, \|u\|_Y\}$ so these are equivalent. We put
$X = \bigcap_{m=5}^\infty W^{m,1}_\sigma(\Omega) \cap W^{m,2}_\sigma(\Omega),$
$X' = \bigcap_{m=5}^\infty W^{m,1}(\Omega) \cap W^{m,2}(\Omega)$.
We define for any $u \in X$,
$\|u\|_X = \sum_{m=5}^\infty \frac{1}{m!^5} \|u\|_{W^{m,1}(\Omega) \cap W^{m,2}(\Omega)}$,
for any $u \in X'$,
$\|u\|_{X'} = \sum_{m=5}^\infty \frac{1}{m!^5} \|u\|_{W^{m,1}(\Omega) \cap W^{m,2}(\Omega)}$.
For a constant $M > 0$, let S be a subset of X :
$S = \{u \in X : \|u\|_X \le M\}$. Let the fundamental solution of $\partial_t - \Delta$ be E . That is, in the sence that $\mathbb{R}^3$ -valued distribution,

$(\partial_t - \Delta)E(t,x) = \delta(t,x) = \delta(t) \otimes \delta(x)$.
Here,
$$E^i(t,x) = \begin{cases} \frac{1}{\sqrt{4\pi t}^3} e^{-\frac{|x|^2}{4t}} & (t > 0) \\ 0 & (t \leq 0) \end{cases} .$$
(END)

Existence of elementary weak solutions

[Assumptions]
Let the domain Ω be a bounded open set contained in $\mathbb{R} \times \mathbb{R}^3$, have smooth boundary, and satisfy $(0,0) \in \Omega$. We assume that for any multi-index α , $d_\alpha = \sup\{|((t,x) - (t',x'))^\alpha| : (t,x),(t',x') \in \Omega\} \geq 1$. Let the external force $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $f \in S$ and $\|f\|_X \leq M^2$.
Let the set of initial values be
$A = \{u(0,\cdot) : u \in S, u(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\, \chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u\cdot\nabla)u)(t-s,x-y))dsdy\}.$
(END)

[Locally solvability of linear partial differential operator with constant coefficients]
Let the fundamental solution of linear partial differential operator with constant coefficients L on $\mathbb{R}^N$ be E . $E \in \mathcal{D}'$ satisfies $LE = \delta$. For $f \in L^1_{\mathrm{loc}}$, one of the weak solutions of the equation $Lu = f$ on $\Omega \Subset \mathbb{R}^N$ is $u = E * \chi_\Omega f \in \mathcal{D}'(\Omega)$.
(END)

[Proof]
For any $\varphi \in \mathcal{D}(\Omega)$,
$\langle L(E * \chi_\Omega f), \varphi\rangle$
$= \pm\langle E * \chi_\Omega f, L\varphi\rangle$
$:= \pm\langle E(x), \langle \chi_\Omega(y) f(y), L\varphi(x+y)\rangle\rangle$
$= \pm\langle \chi_\Omega(x) f(x), \langle E(y), L\varphi(x+y)\rangle\rangle$
$= \langle \chi_\Omega(x) f(x), \langle LE(y), \varphi(x+y)\rangle\rangle$
$= \langle LE(x), \langle \chi_\Omega(y) f(y), \varphi(x+y)\rangle\rangle$
$= \langle \chi_\Omega(y) f(y), \varphi(y)\rangle = \langle f, \varphi\rangle$.
(END)

[Proposition]
$A \neq \emptyset$. Let $a \in A$. Then there are weak solutions $u, \mathfrak{p}$ of the initial value problem on Ω :

$\partial_t u - \Delta u = f - \nabla \mathfrak{p} - (u \cdot \nabla) u$
$\operatorname{div} u = 0$
$u(0, x) = a(x)$,

in the sence that, $u \in S, \mathfrak{p} \in L^2_{\mathrm{loc}}(\Omega)/\{\mathfrak{p}' : \mathfrak{p}' \sim \mathfrak{q} \iff \nabla(\mathfrak{p}' - \mathfrak{q}) = 0\}$ and $u, \mathfrak{p}$ satisfy for any $\varphi \in \mathcal{D}_\sigma(\Omega)$,
$\langle \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \mathfrak{p} - f, \varphi \rangle = 0,$
for any $\varphi \in \mathcal{D}(\Omega)$,
$\langle \operatorname{div} u, \varphi \rangle = -\sum_{j=1}^3 \langle u^j, \partial_{x^j} \varphi \rangle = 0$.

If $f \neq 0$ then $A \not\ni 0, u \neq 0$; if $f = 0$ then $A = \{0\}, u = 0$. $u, \mathfrak{p}$ are measurable functions on $\mathbb{R} \times \mathbb{R}^3$, so $u, \mathfrak{p}$ are time global and $\lim_{t,|x| \to \infty} \partial^\alpha u(t, x) = 0$.
(END)

We proof it later by Banach's fixed point theorem.

[Smoothness of elementary weak solutions]
Solution $(u, \mathfrak{p})$ are C^∞ -functions.

[Proof of smoothness]
m can be arbitrarily large, so the embedding theorem into Hölder space,
"if $\mathbb{N} \ni m - 4/p > 0$ then $W^{m,p}(\Omega) \subset C^{(m-4/p)-1,\varepsilon}(\overline{\Omega})$ for $\varepsilon \in (0, 1)$ ", in the sence of existence of suitable representative elements, u is C^∞ -function.

f is smooth and $\partial_t u + (u \cdot \nabla) u - \Delta u - f = -\nabla \mathfrak{p}$. Because $-\nabla \mathfrak{p}$ is smooth, so $\mathfrak{p}$ is also smooth.
(END)

[X, X' are norm spaces]
$\|u\|_X$ is finity.
[Proof]

By Sobolev embedding theorem, $\partial^\alpha u$ is bounded. From Taylor's formula, $|\partial^\alpha u(t,x)| \leq C_\alpha \Rightarrow C_\alpha = O(\alpha!)\,(|\alpha| \to \infty)$. In fact, $d_\alpha = \sup\{|((t,x)-(t',x'))^\alpha| : (t,x),(t',x') \in \Omega\} \geq 1$ and
$\lim_{|\alpha|\to\infty} C_\alpha d_\alpha/\alpha! \leq c$,
therefore
$C_\alpha \leq c\alpha!$ follows.
(END)

[Completeness]
X, X' are Banach spaces. $\chi_\Omega \in X, \chi_\Omega \in X'$ so $X, X' \neq \{0\}$.
(END)

[Proof]
Let $\{u_n\}$ be a Cauchy sequence in X . Then, for any $m \geq 5$, $\{u_n\}$ is a Caucy sequence of $W^{m,1}_\sigma(\Omega) \cap W^{m,2}_\sigma(\Omega)$. $W^{m,1}_\sigma(\Omega) \cap W^{m,2}_\sigma(\Omega)$ is a Banach space, so $\{u_n\}$ converges. Let the limit be u . $u \in X$. For any positive number ε , there exists a natural number N such that if $\ell, n \geq N$ then
$\|u_\ell - u_n\|_X < \varepsilon$.
From using Fatou's lemma for counting measure,
$\|u - u_n\|_X$
$= \sum_{m=5}^\infty \frac{1}{m!^5} \|u - u_n\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)}$
$= \sum_{m=5}^\infty \frac{1}{m!^5} \liminf_{\ell\to\infty} \|u_\ell - u_n\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)}$
$\leq \liminf_{\ell\to\infty} \sum_{m=5}^\infty \frac{1}{m!^5} \|u_\ell - u_n\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)}$
$\leq \varepsilon$.
(END)

[Separation of product and absorption of differential]
Constants $C_1, C_2 > 0$ exist such that
$\left\|u^i v^i\right\|_{X'} \leq C_1 \|u^i\|_{X'} \|v^i\|_{X'}$
(separation of product)
holds for any $u, v \in X'$, and
$\left\|\partial_{x^j} u\right\|_X \leq C_2 \|u\|_X$
(absorption of differential)
holds for any $u \in X$.
(END)

[Proof]
For the binomial coefficients $c_{\alpha,\beta}$, let
$c_\alpha = \sum_{\beta \le \alpha} c_{\alpha,\beta}$.
There is a continuous embedding $X' \subset C^{k,\varepsilon}(\overline{\Omega})$ for any natural number k , because $\|u_n - u\|_X \to 0$
$\Rightarrow \|u_n - u\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)} \to 0$
$\Rightarrow \|u_n - u\|_{C^{k,\varepsilon}(\overline{\Omega})} \to 0$, so there exists a constant $c' > 0$ such that $\|u\|_{C^{k,\varepsilon}(\overline{\Omega})} \le c'\|u\|_{X'}$. If $|\alpha| \le k$, by Leibniz' formula,
$\|\partial^\alpha(u^i v^i)\|_{L^p(\Omega)}$
$\le c_\alpha \|u^i\|_{C^{k,\varepsilon}(\overline{\Omega})}\|v^i\|_{C^{k,\varepsilon}(\overline{\Omega})}|\Omega|^{1/p}$
$\le c_\alpha c'|\Omega|^{1/p}\|u^i\|_{X'}c'\|v^i\|_{X'}$
$\le c_\alpha c'^2|\Omega|^{1/p}\|u^i\|_{X'}\|v^i\|_{X'}$. Therefore,
$\|\partial^\alpha(u^i v^i)\|_{L^p(\Omega)} \le c_\alpha c'^2|\Omega|^{1/p}\|u^i\|_{X'}\|v^i\|_{X'}$, so there exists a constant $C_1 > 0$ such that
$\|u^i v^i\|_{X'} \le C_1\|u^i\|_{X'}\|v^i\|_{X'}$.

Let $\{u_n\} \subset X$ satisfies $u_n \to u, \partial_{x^j}u_n \to v$. From Hölder's inequality, we have
$|\langle \partial_{x^j}u_n - v, \varphi\rangle|$
$\le \|\partial_{x^j}u_n - v\|_{L^p(\Omega)}\|\varphi\|_{L^q(\Omega)}$
$\to 0\,(p = 1 \Rightarrow q = \infty, p = 2 \Rightarrow q = 2)$ and the weak differentiation is continuous in $\mathcal{D}'_\sigma(\Omega)$, so $\partial_{x^j}u_n \to \partial_{x^j}u \,\mathrm{in}\, \mathcal{D}'_\sigma(\Omega)$. From
$v = \partial_{x^j}u \in X$, $\{u \in X : \partial_{x^j}u \in X\} = X$, the absorption of differential is true by the closed graph theorem.

Or
$\|\partial_{x^j}u\|_X = \sum_{m=5}^\infty \frac{1}{m!^5}\|\partial_{x^j}u\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)}$
$\le \sum_{m=4}^\infty \frac{1}{m!^5}\|u\|_{W^{m+1,1}(\Omega)\cap W^{m+1,2}(\Omega)}$
$\le C_2\sum_{m=5}^\infty \frac{1}{m!^5}\|u\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)}$.
(END)

[Boundness of $X \ni u \mapsto E * (\chi_\Omega u) \in X$]
$X \ni u \mapsto E * (\chi_\Omega u) \in X$ is a bounded operator, so constant $C_3 > 0$ exists such that for any $u \in X$,
$\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)u(t-s,x-y)dsdy\|_X$
$\le C_3\|u\|_X$

holds.
(END)

[Proof]
As a function of (s,y) , for any $(t,x) \in \Omega$,
$\mathrm{supp}(E^i(s,y)\chi_\Omega(t-s,x-y)u^i(t-s,x-y))$
$\subseteq -\overline{\Omega} + (t,x)$
$= \overline{\{(s,y) \in \mathbb{R} \times \mathbb{R}^3 : (t-s,x-y) \in \Omega\}}$
is the translation of reverse of $\overline{\Omega}$, so it is compact, and
$|\partial^\alpha_{t,x}(E^i(s,y)\chi_\Omega(t-s,x-y)u^i(t-s,x-y))| \leq E^i(s,y)\sup\{|\partial^\alpha_{t,x}u^i(t-s,x-y)| : (t,x) \in \Omega\} \leq C_\alpha E^i(s,y) \in L^1_{s,y}(\Omega)$, so combine the theorem of differentiation under the integral sign, Hölder's inequality and continues embedding $X \subset L^\infty(\Omega)$,
$\|\partial^\alpha(E*(\chi_\Omega u))\|_{L^p(\Omega)}$
$\leq \|E*(\partial^\alpha(\chi_\Omega u))\|_{L^p(\Omega)}$
$\leq \|\|E(s,y)\|_{L^1_{s,y}(-\Omega+(t,x))}\|\partial^\alpha u(t-s,x-y)\|_{L^\infty_{s,y}(-\Omega+(t,x))}\|_{L^p_{t,x}(\Omega)}$
$\leq \sup\{\|E\|_{L^1(-\Omega+(t,x))} : (t,x) \in \Omega\}\|\partial^\alpha u\|_{L^\infty(\Omega)}|\Omega|^{1/p}$
$\leq \sup\{\|E\|_{L^1(-\Omega+(t,x))} : (t,x) \in \Omega\}c''C_2^{|\alpha|}\|u\|_X|\Omega|^{1/p}$
$< \infty$.
So we have
$\|E*(\chi_\Omega u)\|_X \leq C_3\|u\|_X.$
(END)

We take $C = \max\{C_1, C_2, C_3\}$. The separation of product, the absorption of differential, and the boundness of $X \ni u \mapsto E*(\chi_\Omega u) \in X$ hold for C .

We solve
(N-S)' $\partial_t u - \Delta u = f - (u\cdot\nabla)u$,
that is, for any $a \in A$, there exist $u \in S, \mathfrak{p} \in L^2_{\mathrm{loc}}(\Omega)/\{\mathfrak{p}' : \mathfrak{p}' \sim \mathfrak{q} \iff \nabla(\mathfrak{p}'-\mathfrak{q}) = 0\}$, such that for any $\varphi \in \mathcal{D}_\sigma(\Omega)$,
$\langle \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\mathfrak{p} - f, \varphi\rangle = 0,$
for any $\varphi \in \mathcal{D}(\Omega)$,
$\langle \mathrm{div}\, u, \varphi\rangle = -\sum_{j=1}^3 \langle u^j, \partial_{x^j}\varphi\rangle = 0$,
$u(0,x) = a(x)$.

$\Phi : S \to S$ can be defined as
$\Phi[u](t,x)$
$= \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u\cdot\nabla)u)(t-s,x-y))dsdy$. We take a function sequence $\{u_n\} \subset S$ as $u_0 : \Omega \to \mathbb{R}^3$ and $u_0 \in S$, if $n \geq 0$ then $u_{n+1}(t,x) = \Phi[u_n](t,x)$
$= \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u_n\cdot\nabla)u_n)(t-s,x-y))dsdy$. If X is a complete metric space, then S is complete because it is a closed subset that is not empty, and if Φ is a contraction mapping, according to the Banach's fixed point theorem, the uniqueness and the existence of a fixed point of Φ follows:

Some $u \in S$ exists uniquely and $\Phi[u] = u$.

Then, due to the uniqueness of the fixed point in Banach's fixed point theorem, u is an unique weak solution. If $f \neq 0$ then $A \not\ni 0, u \neq 0$. Let $f = 0$. From the properties of X , if $u \in X$ and $u(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u\cdot\nabla)u)(t-s,x-y))dsdy$ then $\|u\|_X \leq 3C^3\|u\|_X^2$. So, if $u \neq 0$ then $1 \leq 3C^3\|u\|_X$. By $C = O(|\Omega|)\ (|\Omega| \to 0)$ and the absolute continuity of Lebesgue integral, $|\Omega| \to 0 \Rightarrow C \to 0, \|u\|_X \to 0$, therefore $f = 0 \Rightarrow A = \{0\}, u = 0$. $u, \mathfrak{p}$ are measurable functions on $\mathbb{R}\times\mathbb{R}^3$, so $u, \mathfrak{p}$ are time global.

[Proof of the possibility that Φ can be defined as a contraction mapping]
$u \in S \Rightarrow \|E * (\chi_\Omega(Pf - P((u\cdot\nabla)u)))\|_X < \infty$
holds. Therefore
$\|\Phi[u]\|_X \leq M.$

$\|P\| = 1$, so
$\|\chi_\Omega(Pf - P((u\cdot\nabla)u))\|_X$
$\leq \|f\|_X + \|u^1\partial_{x^1}u + u^2\partial_{x^2}u + u^3\partial_{x^3}u\|_X$
$\leq M^2 + 3C^2M^2 < \infty$.

If
$\|\Phi[u]\|_X$
$\leq CM^2 + 3C^3M^2$
$\leq M$, M must be $C(1+3C^2)M \leq 1$.

(END)

$\Phi : S \to S$ may be Lipschitz continuous: there may be a constant $L > 0$ such that
$\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v)(t-s,x-y) - P((u\cdot\nabla)u)(t-s,x-y))dsdy\|_X$
$\leq L\|u-v\|_X$.

If the Lipschitz continuity established,

$\| \Phi[u] - \Phi[v] \|_X$
$\leq \| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v)(t-s,x-y) - P((u\cdot\nabla)u)(t-s,x-y))dsdy\|_X$
$\leq L\|u-v\|_X$
follows. Here,

[Φ may be a contraction mapping]
$L < 1$

holds.

[Proof of Lipschitz continuity]
$(v\cdot\nabla)v(t-s,x-y) - (u\cdot\nabla)u(t-s,x-y)$
$= \sum_{j=1}^3 (v^j(\partial_{x^j}v(t-s,x-y) - \partial_{x^j}u(t-s,x-y)) + (v^j\partial_{x^j}u(t-s,x-y)) - (u^j\partial_{x^j}u(t-s,x-y)))$, so we have

$\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v)(t-s,x-y) - P((u\cdot\nabla)u)(t-s,x-y))dsdy\|_X$
$\leq C^2\|v\|_X \max_j(\|\partial_{x^j}v - \partial_{x^j}u\|_X) + C^2\|v-u\|_X \max_j(\|\partial_{x^j}u\|_X)$
$\leq C^3M\|v-u\|_X + C^3M\|v-u\|_X$
$= 2C^3M\|u-v\|_X.$

Therefore, Lipschitz continuity follows for $L = 2C^3M$.
(END)

[Proof of the possibility that $\varPhi$ is a contraction mapping]
From the above argment
$\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v(t-s,x-y))-P((u\cdot\nabla)u)(t-s,x-y))dsdy\|_X$
$\le 2C^3M\|u-v\|_X$
and
$2C^3M<1.$
(END)

[Solvability of the Navier-Stokes equations]
The fixed point u of $\varPhi : S \to S$ is the solution of (N-S)'.

[proof]
u satisfies $\operatorname{div} u = 0$ in the sense of a distribution belonging to $\mathcal{D}'(\Omega)$. That is, for any $\varphi \in \mathcal{D}(\Omega)$, $\langle \operatorname{div} u, \varphi\rangle = -\sum_{j=1}^3 \langle u^j, \partial_{x^j}\varphi\rangle = 0$.
In fact, for any $u \in W_\sigma^{m,p}(\Omega)$ there exists a Cauchy sequence $\{u_n\} \subset V_\sigma^{m,p}(\Omega)$, by the integration by parts and Hölder's inequality, we have
$0 = -\sum_{j=1}^3 \langle u_n^j, \partial_{x^j}\varphi\rangle$
$\to -\sum_{j=1}^3 \langle u^j, \partial_{x^j}\varphi\rangle$.

For any $\varphi \in \mathcal{D}_\sigma(\Omega)$,
$\operatorname{div}(\varphi) = 0$, so by the integration by parts
$\langle \nabla \mathfrak{p}, \varphi\rangle$
$= \int_\Omega \sum_{i=1}^3 (\nabla\mathfrak{p})^i(t,x)\varphi^i(t,x)dtdx$
$= -\int_\Omega \mathfrak{p}(t,x)\operatorname{div}(\varphi)(t,x)dtdx = 0.$

Therefore, boundness of $u, \partial_{x^j}u$ by Sobolev embedding theorem and $|\Omega| < \infty$, we have $(u\cdot\nabla)u \in L^2(\Omega)$, so by Helmholtz decomposition,
if we let $f = Pf + \nabla\mathfrak{f}, (u\cdot\nabla)u = P((u\cdot\nabla)u) + \nabla\mathfrak{u}$
then for any $\varphi \in \mathcal{D}_\sigma(\Omega)$,
$\langle f, \varphi\rangle = \langle Pf, \varphi\rangle, \langle\ (u\cdot\nabla)u, \varphi\rangle = \langle P((u\cdot\nabla)u), \varphi\rangle$, hence we solve

(N-S)' $\partial_t u - \Delta u = f - (u\cdot\nabla)u \,\mathrm{in}\, \mathcal{D}_\sigma'(\Omega)$.

By the locally solvability, the solution of the approximate equation on Ω

(N-S)'' $\partial_t v_n - \Delta v_n = Pf - P((u_n \cdot \nabla)u_n)$
satisfies
$v_n = u_{n+1} = E * \chi_\Omega (Pf - P((u_n \cdot \nabla)u_n))$.

Therefore, the solution of (N-S)'' satisfies
$u_{n+1}(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u_n \cdot \nabla)u_n)(t-s,x-y))dsdy.$

$\partial_t u_{n+1}(t,x) - \Delta u_{n+1}(t,x)$
$= \langle (\partial_t E(t-s,x-y) - \Delta E(t-s,x-y)), \chi_\Omega(s,y)(Pf(s,y) - P((u_n \cdot \nabla)u_n)(s,y)) \rangle$
$= \langle \delta(\tau) \otimes \delta(z), \chi_\Omega(t-\tau,x-z)(Pf(t-\tau,x-z) - P((u_n \cdot \nabla)u_n)(t-\tau,x-z)) \rangle$
$= Pf(t,x) - P((u_n \cdot \nabla)u_n)(t,x).$

Therefore, the above calculation and the continuity of the heat operator on $\mathcal{D}'_\sigma(\Omega)$:
$|\langle \partial_t u_{n+1} - \Delta u_{n+1}, \varphi \rangle - \langle \partial_t u - \Delta u, \varphi \rangle| \to 0$, and from Hölder's inequality, $\|P\| = 1$, and product of the functions $L^2(\Omega) \times L^2(\Omega) \ni (u,v) \mapsto uv \in L^1(\Omega)$ is continuous (see [Supplement 2]), so
$|\int_\Omega (P((u_n \cdot \nabla)u_n)(t,x)$
$-P((u \cdot \nabla)u)(t,x))) \cdot \varphi(t,x)dtdx|$
$\le \|((u_n \cdot \nabla)u_n)(t,x) - ((u \cdot \nabla)u)(t,x)\|_{L^1(\Omega)} \|\varphi(t,x)\|_{L^\infty(\Omega)} \to 0 \, (n \to \infty)$,
hence
$\partial_t u - \Delta u = Pf - P((u \cdot \nabla)u)$ holds, so we have
$u(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u \cdot \nabla)u)(t-s,x-y))dsdy$.

u is a solution in the sence of distribution in $\mathcal{D}'_\sigma(\Omega)$ of (N-S)' .

For any $U \in \mathcal{D}'_\sigma(\Omega)$,
" $\varphi \in \mathcal{D}_\sigma(\Omega) \Rightarrow \langle U, \varphi \rangle = 0$ "
$\iff$ " there exists a distribution $\mathfrak{p}$ such that $U = \nabla \mathfrak{p}$ "

by Helmholtz decomposition, therefore there exists $\mathfrak{p}$ such that $\partial_t u + (u \cdot \nabla)u - \Delta u - f = -\nabla \mathfrak{p}$ holds.
(END)

Properties of the solutions

[Continuity of $f \mapsto u, f \mapsto \mathfrak{p}$]
Let $f_n, f \in S, \|f_n\|_X, \|f\| \le M^2, \|f_n - f\|_X \to 0$. Let the solutions be $u_n, \mathfrak{p}_n$ for f_n and $a_n \in A$, let the solutions be $u, \mathfrak{p}$ for f and $a \in A$. Then
$\|u_n - u\|_X \to 0,$
$d(\mathfrak{p}_n, \mathfrak{p}) := \|u_n - u\|_X \to 0$.
(END)

[Proof]
$\|u_n - u\|_X = \| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\, \chi_\Omega(t-s, x-y)(\, Pf_n(t-s, x-y) - P((u_n \cdot \nabla)u_n)(t-s, x-y))dsdy - \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\, \chi_\Omega(t-s, x-y)(\, Pf(t-s, x-y) - P((u \cdot \nabla)u)(t-s, x-y))dsdy\|_X$
$\le C\|f_n - f\|_X + 2C^3 M\|u_n - u\|_X$.
So
$\limsup_{n\to\infty} \|u_n - u\|_X$
$\le 2C^3 M \limsup_{n\to\infty} \|u_n - u\|_X$.

$\limsup_{n\to\infty} \|u_n - u\|_X \le 2M$,
therefore
$0 \le (1 - 2C^3 M) \limsup_{n\to\infty} \|u_n - u\|_X$
≤ 0 .
Hence
$\limsup_{n\to\infty} \|u_n - u\|_X$
$= \lim_{n\to\infty} \|u_n - u\|_X = 0$.
There exist the maps $f \mapsto u, u \mapsto \mathfrak{p}$ so $d(\mathfrak{p}_n, \mathfrak{p}) := \|u_n - u\|_X \to 0$.
(END)

[Continuity of $a \mapsto u, a \mapsto \mathfrak{p}$]
Let the solutions be $u_a, v_b, \mathfrak{p}_a, \mathfrak{q}_b$ for a, b . If we define the metrics given by
$d_A(a,b) = \|u_a - v_b\|_X, D(\mathfrak{p}, \mathfrak{q}) = \|u_a - v_b\|_X$, then $a \mapsto u, a \mapsto \mathfrak{p}$ are continuous.

(END)

[Vanishing]
$\lim_{t,|x|\to\infty} \partial^\alpha u(t,x) = 0$.
(END)

[Proof]
u is a measurable function on $\mathbb{R} \times \mathbb{R}^3$, so we can take the limits as $t, |x| \to \infty$.

$\partial^\alpha u(t,x) = \int_\Omega E(t-s, x-y)\, \partial^\alpha (Pf(s,y) - P((u\cdot\nabla)u)(s,y)) dsdy$, for any $t_0 > 0$, if $t-s > t_0$ then
$|E^i| \le 1/t_0^{3/2},$
$\partial^\alpha (Pf - P((u\cdot\nabla)u)) \in X \subset C^{0,\varepsilon}(\overline{\Omega})$
so $\lim_{t,|x|\to\infty} \partial^\alpha u(t,x) = 0$ follows from the bounded convergence theorem.
(END)

From the properties of X and $u = \Phi[u]$,
$\|u\|_X \le C\|f\|_X + 3C^3\|u\|_X^2 \le M$.
$CM \le C(1+3C^2)M \le 1$
so
$C\|f\|_X \le CM^2 \le M$.
Therefore, from
$C\|f\|_X + 3C^3\|u\|_X^2 \le M$, we have
$\|u\|_X \le \sqrt{\frac{M - C\|f\|_X}{3C^3}} < M$.

Supplements

[Supplement 1]
As functions φ that $\mathrm{div}\varphi = \nabla\cdot\varphi = 0$, it is sufficient to take any $\psi \in \mathcal{D}(\Omega)$ and set to $\varphi = \mathrm{curl}\psi$.

[Supplement 2]
Let $\|u_n - u\|_{L^2(\Omega)} \to 0, \|v_n - v\|_{L^2(\Omega)} \to 0$. By the triangle inequality, we have $|\|u_n\|_{L^2(\Omega)} - \|u\|_{L^2(\Omega)}| \le \|u_n - u\|_{L^2(\Omega)}$ for any sufficientaly large n . On the other hand,

$\|u_n\|_{L^2(\Omega)} < \|u\|_{L^2(\Omega)} + 1$. Therefore

$$\|u_n v_n - uv\|_{L^1(\Omega)} \leq \|u_n\|_{L^2(\Omega)}\|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|u_n - u\|_{L^2(\Omega)}$$
$$< (\|u\|_{L^2(\Omega)} + 1)\|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|u_n - u\|_{L^2(\Omega)} \to 0.$$

References

[1]俣野博-神保道夫, 熱・波動と微分方程式, 岩波オンデマンドブックス, 岩波書店, 2018

[2]金子晃, 偏微分方程式入門, 基礎数学12, 東京大学出版会, 2013

[3]垣田高夫-柴田良弘, ベクトル解析から流体へ, 日本評論社, 2007

[4]谷島賢二, 数理物理入門 改訂改題, 基礎数学11, 東京大学出版会, 2018

[5]柴田良弘, ルベーグ積分論, 内田老鶴圃, 2006

[6]谷島賢二, 新版 ルベーグ積分と関数解析, 講座〈数学の考え方〉13, 朝倉書店, 2015

[7]コルモゴロフ-フォミーン, 函数解析の基礎 上, 岩波書店, 2012

[8]北田均, 新訂版 数理解析学概論, 現代数学社, 2016

[9]猪狩惺, 実解析入門, 岩波書店, 2013

[10]岡本久-中村周, 関数解析, 岩波オンデマンドブックス, 岩波書店, 2016

[11]黒田成俊, 関数解析, 共立数学講座15, 共立出版, 2011

[12]藤田宏-黒田成俊-伊藤清三, 関数解析, 岩波書店, 2009

[13]Kôsaku Yosida, Functional Analysis, Springer-Verlag, 1980

[14]増田久弥, 応用解析ハンドブック, 丸善出版, 2012

[15]溝畑茂, 偏微分方程式論, 岩波書店, 2010

[16]小薗英雄-小川卓克-三沢正史, これからの非線型偏微分方程式, 日本評論社, 2007

[17]垣田高夫, シュワルツ超関数入門, 日本評論社, 2015

[18]宮島静雄, ソボレフ空間の基礎と応用, 共立出版, 2020

[19]澤野嘉宏, ベゾフ空間論, 日本評論社, 2011

[20]岡本久, ナヴィエ-ストークス方程式の数理 新装版, 東京大学出版会, 2023

[21]柴田良弘, 流体数学の基礎 上, 岩波数学叢書, 岩波書店, 2022

[22]柴田良弘, 流体数学の基礎 下, 岩波数学叢書, 岩波書店, 2022

[23]Lars Hörmander, The Analysis of Linear Partial Differential Operators I: Distribution Theory And Fourier Analysis, Springer, 1990

[24]柴田良弘-久保隆徹, 非線形偏微分方程式, 現代基礎数学21, 朝倉書店, 2013

[25]八木厚志, 放物型発展方程式とその応用(上) 可解性の理論, 岩波数学叢書, 岩波書店, 2011

[26]小川卓克, 非線型発展方程式の実解析的方法, シュプリンガー現代数学シリーズ 第18巻, 丸善出版, 2013

[27]新井仁之, ルベーグ積分講義 ルベーグ積分と面積0の不思議な図形たち, 日本評論社, 2003

[28]Adams-Fournier, Sobolev Spaces, Academic Press, 2003

[29]Wasao SIBAGAKI, Hisako RIKIMARU, ON THE E. HOPF'S WEAK SOLUSION OF INITIAL VALUE PROBLEM FOR THE NAVIER-STOKES EQUATIONS, 1967

[30]Ping Zhang, Global Fujita-Kato solution of 3-D inhomogeneous incompressible Navier-Stokes system, Advances in Mathematics 363, 107007, 2020

[31]Frédéric Charve, Raphaël Danchin, A Global Existence Result for the Compressible Navier–Stokes Equations in the Critical L^p Framework, Archive for Rational Mechanics and Analysis 198 (1), 233-271, 2010

[32]Yoshihiro Shibata, Miho Murata, On the global well-posedness for the compressible Navier–Stokes equations with slip boundary condition, Journal of Differential Equations 260 (7), 5761-5795, 2016