A Geometric Algebra Solution to a Hard Contest Problem

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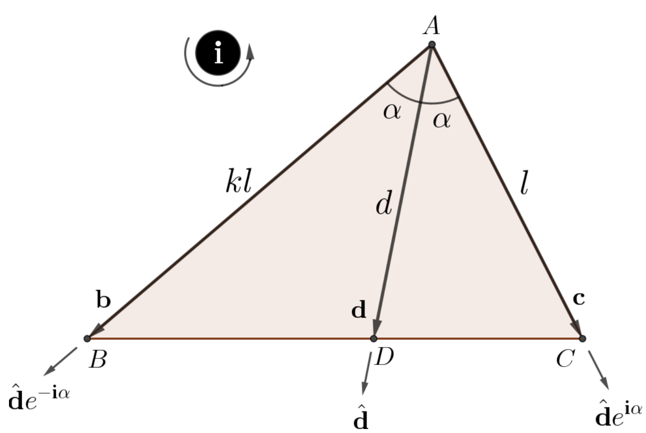
Abstract

We show how to use rotations of vectors in GA to solve the following problem: "The following are known about a triangle: The ratio of the lengths of two sides; the angle formed by those sides; and the length of that angle's bisector. Find the length of the side opposite that angle."



Given α, d, and k (but not l), find the length BC .

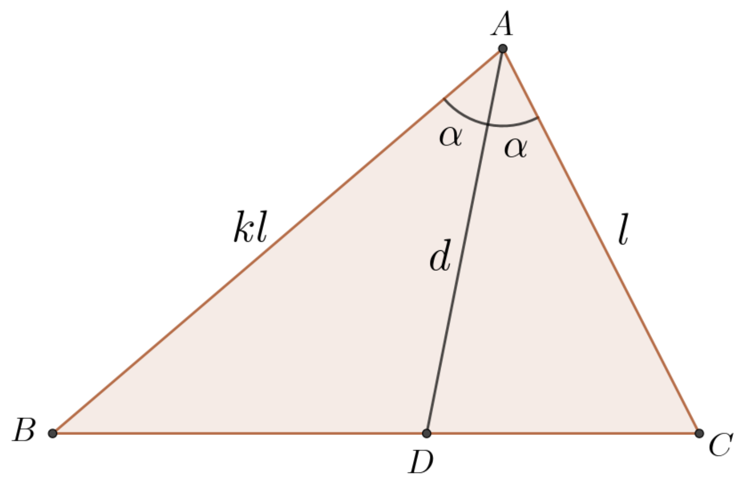


Figure 1: Given α, d, and k (but not l), find the length BC .

1 Statement of the Problem

The problem that we solve here is a generalization of a contest problem that the YouTube channel "Mind Your Decisions" solved by conventional means at https://www.youtube.com/watch?v=BeuLmUjPFsk. We state that generalizaton as: Given α, d, and k (but not l), find the length BC (Fig. 1).

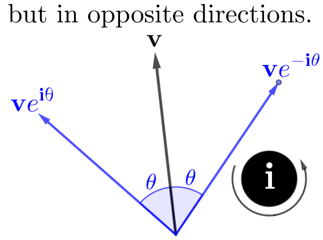
2 Ideas that We Will Use

1. For any two parallel vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \wedge \mathbf{v} = 0$.

2. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \wedge \mathbf{v} = \langle \mathbf{uv} \rangle_2$.

3. Ideas concern the rotation of a vector $\mathbf{v}$ that is parallel to a given bivector $\mathbf{i}$:

   (a) The vectors $\mathbf{v}e^{\mathbf{i}\theta}$ and $\mathbf{v}e^{-\mathbf{i}\theta}$ are rotations of $\mathbf{v}$ by the same angle θ, but in opposite directions.

   (b) For any angle θ, and any vector $\mathbf{v}$ that is parallel to bivector $\mathbf{i}$, $\mathbf{v}e^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}\mathbf{v}$. Here is a proof:

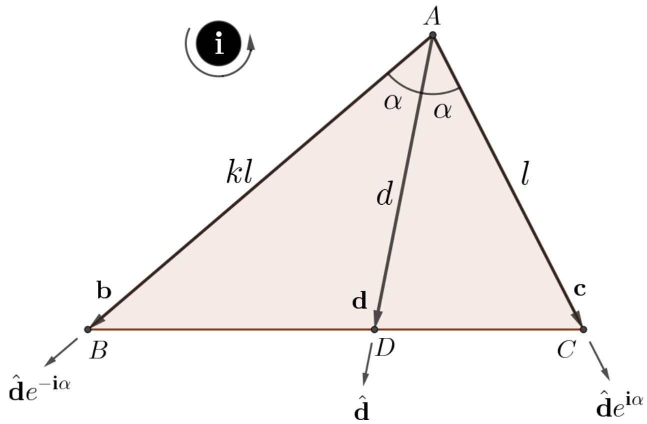


Figure 2: Formulation of the problem in terms of vectors. Note that $\mathbf{b} = kl\hat{\mathbf{d}}e^{-\mathbf{i}\alpha}$; $\mathbf{d} = d\hat{\mathbf{d}}$; and $\mathbf{c} = l\hat{\mathbf{d}}e^{\mathbf{i}\alpha}$.

$$\begin{aligned}
\mathbf{v}e^{\mathbf{i}\theta} &= \mathbf{v}\left[\cos\theta + \mathbf{i}\sin\theta\right] \\
&= \mathbf{v}\cos\theta + \mathbf{v}\mathbf{i}\sin\theta \\
&= \mathbf{v}\cos\theta - \mathbf{i}\mathbf{v}\sin\theta \\
&= \left[\cos\theta - \mathbf{i}\sin\theta\right]\mathbf{v} \\
&= e^{-\mathbf{i}\theta}\mathbf{v}.
\end{aligned}$$

(c) From $\mathbf{v}e^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}\mathbf{v}$, we can see that $\mathbf{v}e^{\mathbf{i}\theta}\mathbf{v} = v^2e^{-\mathbf{i}\theta}$, and that $\hat{\mathbf{v}}e^{\mathbf{i}\theta}\hat{\mathbf{v}} = e^{-\mathbf{i}\theta}$.

3 Solution Strategy

We will use rotation of vectors to determine l, then (knowing l) we will use the Law of Cosines to calculate BC.

4 Formulation in Terms of Vectors

We formulate the problem as shown in Fig. 2.

5 Solution

We begin by recognizing that $[\mathbf{c}-\mathbf{d}] \parallel [\mathbf{d}-\mathbf{b}]$. Therefore, $[\mathbf{c}-\mathbf{d}] \wedge [\mathbf{d}-\mathbf{b}] = 0$, and

$$\begin{aligned}
\langle [\mathbf{c}-\mathbf{d}]\,[\mathbf{d}-\mathbf{b}] \rangle_2 &= 0 \\
\langle \left[l\hat{\mathbf{d}}e^{\mathbf{i}\alpha} - d\hat{\mathbf{d}} \right] \left[d\hat{\mathbf{d}} - kl\hat{\mathbf{d}}e^{-\mathbf{i}\alpha} \right] \rangle_2 &= 0 \\
\langle ld\hat{\mathbf{d}}e^{\mathbf{i}\alpha}\hat{\mathbf{d}} - kl^2\hat{\mathbf{d}}e^{\mathbf{i}\alpha}\hat{\mathbf{d}}e^{-\mathbf{i}\alpha} - d^2 + kdle^{-\mathbf{i}\alpha} \rangle_2 &= 0 \\
\langle lde^{-\mathbf{i}\alpha} - kl^2e^{-\mathbf{i}\alpha}e^{-\mathbf{i}\alpha} + kdle^{-\mathbf{i}\alpha} \rangle_2 &= 0 \\
\langle de^{-\mathbf{i}\alpha} - kle^{-2\mathbf{i}\alpha} + kde^{-\mathbf{i}\alpha} \rangle_2 &= 0 \\
-d\sin\alpha + kl\sin 2\alpha - kd\sin\alpha &= 0 \\
\therefore \quad l &= \frac{(k+1)\,d\sin\alpha}{k\sin 2\alpha}.
\end{aligned}$$

$\hat{\mathbf{d}}e^{\mathbf{i}\alpha}\hat{\mathbf{d}} = e^{-\mathbf{i}\alpha}$.
$\langle d^2 \rangle_2 = 0$.

Because $\sin 2\alpha = 2\sin\alpha\cos\alpha$,

$$l = \frac{(k+1)\,d}{2k\cos\alpha}.$$

Finally, from the Law of Cosines,

$$\begin{aligned}
BC &= \sqrt{AB^2 + AC^2 - 2\,(AB)\,(AC)\cos 2\alpha} \\
&= \sqrt{(kl)^2 + l^2 - 2kl^2\cos 2\alpha} \\
&= l\sqrt{k^2 + 1 - 2k\cos 2\alpha} \\
&= \left[\frac{(k+1)\,d}{2k\cos\alpha}\right]\sqrt{(k+1)^2 - 2k\,(1+\cos 2\alpha)} \\
&= \left[\frac{(k+1)\,d}{2k\cos\alpha}\right]\sqrt{(k+1)^2 - 4k\cos^2\alpha}\ .
\end{aligned}$$

For any angle θ, $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] J. A. Smith, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", 2016.