# DECIMALISING INTEGER SEQUENCES

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Abstract. In this paper, we observe how some well-known integer sequences when divided by powers of 10 and summed to infinity have a unique discrete value, similar to a person's 'DNA'.

In this paper we observe how certain integer sequences when divided by powers of 10 and summed to infinity have discrete values. We will look at the Triangular numbers, $T_{(n)}$, the Tetrahedral numbers, $Te_{(n)}$, Lazy Caterer's numbers, $LC_{(n)}$, the Fibonacci numbers, $F_{(n)}$, the Lucas numbers, $L_{(n)}$, the Pell numbers, $P_{(n)}$, the first differences of the Pell numbers, $P^1_{(n)}$, the Jacobsthal numbers, $J_{(n)}$, Padovan numbers, $Pa_{(n)}$, and Narayana's Cows numbers, $NC_{(n)}$, the Catalan numbers, $C_{(n)}$, the Central Binomial Coefficients, $CB_{(n)}$.

We can summarise our findings as follows:

$$\sum_{n=1}^{\infty} \frac{T_{(n)}}{10^n} = \frac{10^2}{9^3}$$

$$\sum_{n=1}^{\infty} \frac{LC_{(n)}}{10^n} = \frac{91}{9^3}.$$

$$\sum_{n=1}^{\infty} \frac{Te_{(n)}}{10^n} = \frac{10^2}{9^4}$$

$$\sum_{n=1}^{\infty} \frac{F_{(n)}}{10^n} = \frac{1}{89}$$

$$\sum_{n=1}^{\infty} \frac{L_{(n)}}{10^n} = \frac{19}{89}$$

$$\sum_{n=1}^{\infty} \frac{P_{(n)}}{10^n} = \frac{1}{79}$$

$$\sum_{n=1}^{\infty} \frac{P^1_{(n)}}{10^n} = \frac{9}{79}$$

$$\sum_{n=1}^{\infty} \frac{J_{(n)}}{10^n} = \frac{161}{1100}$$

$$\sum_{n=1}^{\infty} \frac{Pa_{(n)}}{10^n} = \frac{110}{989}$$

$$\sum_{n=1}^{\infty} \frac{NC_{(n)}}{10^n} = \frac{100}{899}$$

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$$\sum_{n=1}^{\infty} \frac{E_{(n)}}{10^n} = \frac{5^2}{2.9^2}.$$

$$\sum_{n=1}^{\infty} \frac{C_{(n)}}{10^n} = \frac{\sqrt{5}(\sqrt{5}-\sqrt{3})}{10^2}$$

$$\sum_{n=1}^{\infty} \frac{CB_{(n)}}{10^n} = \frac{1}{\sqrt{96}}.$$

**Triangular Numbers.** The Triangular Numbers, $T_{(n)}$, are defined as:

$$T_{(n)} = \frac{n(n+1)}{2}.$$

The first few Triangular Numbers are as follows:
0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, (OEIS A000217).

We can arrange the Triangular Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{T_{(n)}}{10^n} = 0.1 + 0.03 + 0.006 + 0.0010 + 0.00015 + ...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{T_{(n)}}{10^n} = 0.\overline{137174211248285322359396433470507544581618655692729766803840879149519890260631}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{T_{(n)}}{10^n} = \frac{10^2}{9^3}.$$

As an aside, this reciprocal of 729 has the interesting property of being calculable in reverse, working backwards from the last digit, such that:

$$\begin{array}{l} 0.0000000000000000001+ \\ 0.000000000000000003 \\ 0.00000000000000006 \\ 0.0000000000000010 \\ 0.000000000000015 \\ 0.00000000000021 \\ 0.0000000000028 \\ 0.000000000036 \\ 0.00000000045 \\ ... \\ =......79149519890260631 \end{array}$$

**Tetrahedral Numbers.** The Tetrahedral Numbers, $Te_{(n)}$, are defined as:

$$Te_{(n)} = \frac{n(n+1)(n+2)}{6}.$$

The first few Tetrahedral Numbers, where $Te_{(1)} = 0$, are as follows:
0, 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, 969, 1140, 1330, 1540, 1771, 2024, 2300, 2600, 2925, 3276, 3654, 4060, 4495, 4960,... (OEIS A000292).

We can arrange the Tetrahedral Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{Te_{(n)}}{10^n} = 0 + 0.01 + 0.004 + 0.0010 + 0.00020 + 0.000035 + 0.0000056 + ...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{Te_{(n)}}{10^n} = 0.\overline{01524157890275872580399329370522786160 64...28959}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{Te_{(n)}}{10^n} = \frac{10^2}{9^4}.$$

In general, for all figurate numbers, $P_r$, where $r$ is the $n^{th}$ row of Pascal's Triangle (e.g. $T_{(n)} = P_{4(n)}$, $Te_{(n)} = P_{5(n)}$), we can say that:

$$\sum_{n=1}^{\infty} \frac{P_{r(n)}}{10^n} = \frac{10^2}{9^{r-1}}.$$

The number of repeating digits in each respective decimalisation is $9^{r-2}$.

**Lazy Caterer's Numbers.** Closely related to the Triangular numbers are the Lazy Caterer's numbers, $LC_{(n)}$. These are defined as $\frac{n(n+1)}{2} + 1$. The first few in this sequence are:

1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172, 191, 211, 232, 254, 277, 301, 326, 352, 379, 407, 436, 466, 497, 529, 562, 596, ... (A000124 OEIS).

We can arrange the Lazy Caterer's Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{LC_{(n)}}{10^n} = 0.1 + 0.02 + 0.004 + 0.0007 + 0.00011 + 0.000016 + 0.0000022...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{LC_{(n)}}{10^n} = 0.\overline{124828532235939643347050754458161865569272976680384087791}$$
$$\overline{495198902606311371742 1}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{LC_{(n)}}{10^n} = \frac{91}{9^3}.$$

Notice that the repeating decimals are the same as the repeating decimals of the Triangular Numbers above, only shifted 8 places to the left.

**The Fibonacci Numbers.** The Fibonacci numbers, $F_n$, may be defined by the recurrence relation: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n > 1$.

The first few Fibonacci Numbers are as follows:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, (OEIS A000045).

We can arrange the Fibonacci Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{F_{(n)}}{10^n} = 0.01 + 0.001 + 0.0002 + 0.00003 + 0.000005 + 0.0000008...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{F_{(n)}}{10^n} = 0.\overline{01123595505617977528089887640449438202247191}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{F_{(n)}}{10^n} = \frac{1}{89}.$$

For any positive integer $x$, we get the generating function for Fibonnaci numbers such that:

$$\sum_{n=1}^{\infty} \frac{F_{(n)}}{x^n} = \frac{x}{x^2 - x - 1}.$$

But we also note that the reciprocal of 89 can also be calculated in reverse, working backwards from the last digit, using the powers of 9, such that:

$$\begin{aligned}
&0.0000000000000000001\\
&0.000000000000000009\\
&0.00000000000000081\\
&0.0000000000000729\\
&0.000000000006561\\
&0.00000000059049\\
&0.0000000531441\\
&0.000004782969\\
&0.00043046721\\
&...\\
&=......40449438202247191
\end{aligned}$$

**The Lucas Numbers.** The Lucas numbers, $L_n$, may be defined by the recurrence relation: $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, for $n > 1$.

The first few Lucas Numbers are as follows:
2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, 15127, 24476, 39603, 64079, 103682, 167761, 271443, 439204, 710647, 1149851, 1860498, 3010349, 4870847, 7881196, 12752043, 20633239, 33385282,... (OEIS A000032).

We can arrange the Lucas Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{L_{(n)}}{10^n} = 0.2 + 0.01 + 0.003 + 0.0004 + 0.00007 + 0.000011...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{L_{(n)}}{10^n} = 0.\overline{2134831460674157303370786515853932584269662 9}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{L_{(n)}}{10^n} = \frac{19}{89}.$$

**The Pell Numbers.** The Pell Numbers may be defined by the recurrence relation: $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$. The first few Pell Numbers, $P_{(n)}$, are as follows:

0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, 1136689, 2744210, 6625109, 15994428, 38613965, 93222358, 225058681,... (A000129 OEIS).

We can arrange the Pell Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{P_{(n)}}{10^n} = 0+0.01+0.002+0.0005+0.00012+0.000029+0.0000070+0.00000169...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{P_{(n)}}{10^n} = 0.\overline{0126582278481}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{P_{(n)}}{10^n} = \frac{1}{79}$$

As with the Fibonnaci numbers the reciprocal of 79 can also be calculated in reverse, working backwards from the last digit, using the powers of 8.

**The first differences of the Pell Numbers.** Let us call this sequence $P^1_{(n)}$, since they are the first differences of the Pell numbers. They are also numerators of continued fraction convergent to $\sqrt{2}$, e.g. $\frac{P^1_{(n)}}{P_{(n)}} => \sqrt{2}$.

This sequence may be defined by the recurrence relation: $P^1_0 = 0$, $P^1_1 = 1$, and $P^1_n = 2P^1_{n-1} + P^1_{n-2}$. The first few in this sequence are as follows:

1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, 47321, 114243, 275807, 665857, 1607521, 3880899, 9369319, 22619537, 54608393, 131836323, 318281039, ... (A001333 OEIS).

We can again arrange these as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{P^1_{(n)}}{10^n} = 0 + 0.1 + 0.01 + 0.003 + 0.0007 + 0.000017 + 0.0000041 + 0.00000099...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{P^1_{(n)}}{10^n} = 0.\overline{1139240506329}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{P^1_{(n)}}{10^n} = \frac{9}{79}$$

$$\sum_{n=1}^{\infty} \frac{P^1_{(n)}}{10^n} = \frac{9}{79}$$

**The Jacobsthal Numbers.** The Jacobsthal Numbers can be defined as $J_{(n)} = J_{(n-1)} + 2J_{(n-2)}$ with $J_{(0)} = 0$, and $J_{(1)} = 1$. The first few Jacobsthal Numbers, $J_{(n)}$, are as follows:

0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, 21845, 43691, 87381, 174763, 349525, 699051, 1398101, 2796203, 5592405, 11184811, 22369621,... (A001045 OEIS).

We can arrange the Jacobsthal Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{J_{(n)}}{10^n} = 0.0 + 0.01 + 0.001 + 0.0003 + 0.00005 + 0.000011 + 0.0000021...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{J_{(n)}}{10^n} = 0.014\overline{63}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{J_{(n)}}{10^n} = \frac{161}{1100}.$$

We can arrange them alternatively in the following way:

$$\sum_{n=1}^{\infty} \frac{J_{(n)}}{10^{2n}} = 0.01 + 0.0001 + 0.000003 + 0.00000005 + 0.0000000011...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{J_{(n)}}{10^{2n}} = 0.\overline{0101030511214386744796928672459082649581733683572438876540715296019397858153162255 0}...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{J_{(n)}}{10^{2n}} = \frac{1}{98.98}.$$

**Padovan Numbers.** The Padovan Sequence can be defined as $Pa_{(n)} = Pa_{(n-2)} + Pa_{(n-3)}$ with $Pa_{(0)} = Pa_{(1)} = Pa_{(2)} = 1$. The first few Padovan Numbers are as follows:

1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, 465, 616, 816, 1081, 1432, 1897, 2513, 3329, 4410, 5842,... (A001045 OEIS).

We can arrange the Padovan Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{Pa_{(n)}}{10^n} = 0.1+0.01+0.001+0.0002+0.00002+0.00003+0.0000004+0.00000005...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{Pa_{(n)}}{10^n} = 0.\overline{1112234580384226491405460060667340748230535894843276036400404448938321536905 9}...$$

$$\sum_{n=1}^{\infty} \frac{Pa_{(n)}}{10^n} = \frac{110}{989}.$$

**Narayana's Cows Numbers.** The Narayana's Cows sequence can be defined as $NC_{(n)} = NC_{(n-1)} + NC_{(n-3)}$ with $NC_{(0)} = NC_{(1)} = NC_{(2)} = 1$. The first few Narayana's Cows Numbers, $NC_{(n)}$, are as follows:

1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, 1873, 2745, 4023, 5896, 8641, 12664, 85626, ... (A000930 OEIS).

We can arrange these as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{NC_{(n)}}{10^n} = 0.1+0.01+0.001+0.0002+0.00003+0.000004+0.0000006+0.00000009...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{NC_{(n)}}{10^n} = 0.1112347052280311457174638720800889877641824249165739...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{NC_{(n)}}{10^n} = \frac{100}{899}.$$

**Eulerian Numbers.** Here, we refer to the second column of the classic version of Euler's triangle used by Comtet (1974). This sequence can be defined as $E_{(n)} = 3E_{(n-1)} - 2E_{(n-2)} + 1$ with $E_{(0)} = 1$, $E_{(1)} = 4$. The first few Eulerian Numbers, $E_{(n)}$, are as follows:

1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, 8178, 16369, 32752, 65519, 131054, 262125, 524268, 1048555, 2097130, 4194281, 8388584,... (A000295 OEIS).

We can arrange these as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{E_{(n)}}{10^n} = 0.1+0.04+0.011+0.0026+0.00057+0.000120+0.0000247+0.00000520...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{E_{(n)}}{10^n} = 0.15432\overline{098765432}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{E_{(n)}}{10^n} = \frac{5^2}{2.9^2}.$$

So far, we have identities that do not involve square roots. In the final two examples, we conjecture that sequences defined by factorials will result in identities with square roots.

**The Catalan Numbers.** The Catalan Numbers are defined as $\frac{(2n)!}{n!(n+1)!}$. The first few Catalan Numbers, $C_{(n)}$, are as follows:

0, 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650,... (A000108 OEIS).

We can arrange the Catalan Numbers as an infinite sum in the following way:

$$\sum_{n=1}^{\infty} \frac{C_{(n)}}{10^n} = 0 + 0.1 + 0.01 + 0.002 + 0.005 + 0.0014 + 0.00042 + 0.000132...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{C_{(n)}}{10^n} = 0.01127016653792583114820734600217600389167078294708409...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{C_{(n)}}{10^n} = \frac{(5-\sqrt{15})}{10^2}.$$

If we set the denominator as $10^{2n-1}$, we get the following:

$$\sum_{n=1}^{\infty} \frac{C_{(n)}}{10^{2n-1}} = 0.1 + 0.001 + 0.00002 + 0.0000005 + 0.000000014 + 0.00000000042...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{C_{(n)}}{10^{2n-1}} = 0.1010205144336438036053185058821721606810503868665974313461 4...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{C_{(n)}}{10^{2n-1}} = (\sqrt{3}-\sqrt{2})^2.$$

**The Central Binomial Coefficients.** The Central Binomial Coefficients, found in the central column of Pascal's Triangle, are defined as $\frac{(2n)!}{n!^2}$. The first few Central Binomial Coefficients, $CB_{(n)}$, are as follows:

1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, 705432, 2704156, 10400600, 40116600, 155117520, 601080390, 2333606220, 9075135300, 35345263800,... (A000984 OEIS).

Again, we can arrange them as an infinite sum:

$$\sum_{n=1}^{\infty} \frac{CB_{(n)}}{10^n} = 0.1 + 0.02 + 0.006 + 0.0020 + 0.00070 + 0.000252...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{CB_{(n)}}{10^n} = 0.12909944487358056283930884665941332036109739017638...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{CB_{(n)}}{10^n} = \frac{1}{\sqrt{60}}$$

If we set the denominator as $10^{2n-1}$, we get the following:

$$\sum_{n=1}^{\infty} \frac{CB_{(n)}}{10^{2n-1}} = 0.10102062072615965754091553503112745474665247811694027 9...$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{CB_{(n)}}{10^{2n-1}} = \frac{1}{\sqrt{96}}$$

**More results.** We also find the following results for the first few exponents of the counting numbers:

$$\sum_{n=1}^{\infty} \frac{n}{10^n} = \frac{10}{3^4}.$$

$$\sum_{n=1}^{\infty} \frac{n^2}{10^n} = \frac{110}{3^6}.$$

$$\sum_{n=1}^{\infty} \frac{n^3}{10^n} = \frac{470}{3^7}.$$

$$\sum_{n=1}^{\infty} \frac{n^4}{10^n} = \frac{7370}{3^9}.$$

$$\sum_{n=1}^{\infty} \frac{n^5}{10^n} = \frac{142870}{3^{11}}.$$

But what is the pattern underlying these results?

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