INTEGER OPTIMIZATION AND P vs NP PROBLEM

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ABSTRACT. We transform NP-complete Problem to the polynomial-time algorithm which would mean that P = NP.

1. **Introduction.** Despite in general, Integer Programming is NP-hard or even incomputable (see, e.g., Hemmecke et al. [10]), for some subclasses of target functions and constraints it can be computed in time polynomial.

A fixed-dimensional polynomial minimization in integer variables, where the objective function is a convex polynomial and the convex feasible set is described by arbitrary polynomials can be solved in time polynomial(see, e.g Khachiyan and Porkolab [11]), see Lenstra [13] as well.

A fixed-dimensional polynomial minimization over the integer variables, where the objective function is a quasiconvex polynomial with integer coefficients and where the constraints are inequalities with quasiconvex polynomials of degree at most ≥ 2 with integer coefficients can be solved in time polynomial in the degrees and the binary encoding of the coefficients(see, e.g., Heinz [8], Hemmecke et al. [10], Lee [12]).

Minimizing a convex function over the integer points of a bounded convex set is polynomial in fixed dimension, according to Oertel et al. [15].

Del Pia and Weismantel [4] showed that Integer Quadratic Programming can be solved in polynomial time in the plane.

It was further generalized for cubic and homogeneous polynomials in Del Pia et al. [5].

We are going to transform well-known NP-complete problem to the polynomial-time integer minimization algorithm. It would mean, that P = NP, since if there is a polynomial-time algorithm for any NP-hard problem, then there are polynomial-time algorithms for all problems in NP (see Garey and Johnson [7], Manders and Adleman [14], Cormen et al. [2]).

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Fortnow in [6] stated: "We call the very hardest NP problems (which include Partition Into Triangles, Clique, Hamiltonian Cycle and 3-Coloring) “NP-complete”, i.e. given an efficient algorithm for one of them, we can find efficient algorithm for all of them and in fact any problem in NP".

2. **Polynomial-time Algorithm. Sliding Tangent.**

Lemma 1 (De Loera et al. [3], Hemmecke et al. [10], Del Pia et al. [5]).
The problem of minimizing a degree-4 polynomial over the lattice points of a convex polygon is NP-hard.

Proof. They use the NP-complete problem AN1 on page 249 of Garey and Johnson [7]. This problem states it is NP-complete to decide whether, given three positive integers a, b, c, there exists a positive integer $x < c$ such that x^2 is congruent with a modulo b. This problem is clearly equivalent to asking whether the minimum of the quartic polynomial function $(x^2 - a - by)^2$ over the lattice points of the rectangle:

$\{ (x,y) \mid 1 \le x \le c-1,\ 1-a \le by \le (c-1)^2 - a \}$ is zero or not. □

According to Del Pia and Weismantel [4], minimization problem, given in the above proof of Lemma 1 is equivalent to the following problem:

$$\min \{ (x^2 - a - by) \text{ subject to } \\ x^2 - a - by \ge 0, \\ 1 \le x \le c-1,\ 1-a \le by \le (c-1)^2 - a,\ x, y \in \mathbf{Z} \}. \tag{1}$$

If $L := \{ (x, y) \in \mathbf{R}^2 \mid x^2 - a - by \ge 0,\ x \ge 0 \}$,
$G := \{ (x, y) \in \mathbf{R}^2 \mid 1 \le x \le c-1,\ 1-a \le by \le (c-1)^2 - a \}$,

problem (1) can be rewritten as follows:

$$\mu := \min \{ (x^2 - a - by) \mid (x, y) \in (L \cap G) \cap \mathbf{Z}^2 \}. \tag{2}$$

If $by_{min} = 1 - a$, $by_{max} = (c-1)^2 - a$, then the above defined rectangle:

$G = \{ (x, y) \in \mathbf{R}^2 \mid 1 \leq x \leq c - 1, \ \ y_{min} \leq y \leq y_{max} \}.$

Note that parabola: $by = bf(x) = x^2 - a, \ x \geq 0$ is a part of the border of set L (the top) and we have:

$bf(1) = 1 - a = by_{min}$, $bf(c - 1) = (c - 1)^2 - a = by_{max}$,
$f(1) = y_{min}$, $f(c - 1) = y_{max}$.

Set L is not convex, as well as the set $L \cap G$ (see Boyd and Vandenberghe [1], Osborne [16]).

The equation of the tangent to the parabola: $by = bf(x) = x^2 - a$, at the point i: $1 \leq i \leq c - 1, i \in \mathbf{Z}, x \in \mathbf{R}$ is given by:

$$by_i(x) = 2i(x - i) + i^2 - a. \tag{3}$$

The segment of this tangent (hypotenuse), which is inside G and having one end $D_i = (d_{1i}, d_{2i})$ on the horizontal line $by = 1 - a$, and another end $H_i = (h_{1i}, h_{2i})$ on the vertical line $x = c - 1$, together with two other segments: on the horizontal line $by = 1 - a$ and on the vertical line $x = c - 1$, both segments intersected at point $E = (e_1, e_2)$: $e_1 = c - 1$, $be_2 = 1 - a$ (cathetuses), form some right triangle D_iH_iE:

$D_iH_iE := S_i := \{ (x, y) \in G \mid y \leq y_i(x) \}$,
$d_{1i} \in \mathbf{R}, d_{2i} \in \mathbf{R}, h_{1i} \in \mathbf{R}, h_{2i} \in \mathbf{R}, \ e_1, e_2, \ 1 \leq i \leq c - 1, i \in \mathbf{Z}.$

Proposition 1. *$2id_{1i} = i^2 + 1$, $bd_{2i} = 1 - a$,*
$h_{1i} = c - 1$, $bh_{2i} = 2i(c - 1) - i^2 - a$,
$1 \leq i \leq c - 1, i \in \mathbf{Z}$.

Proof. It follows from the definition of points D_i, H_i and (3): considering points D_i and H_i as intersections of the tangent (3) and the corresponding horizontal and vertical lines, described above, we have for the points D_i: $y_i(d_{1i}) = d_{2i} = y_{min}$, and for the points H_i: $h_{2i} = y_i(h_{1i}) = y_i(c - 1)$. □

Lemma 2. *$(L \cap G) \cap \mathbf{Z}^2 = \cup (S_i \cap \mathbf{Z}^2)$, $1 \leq i \leq c - 1, i \in \mathbf{Z}$.*

Proof. It follows from the above given definitions and properties of sets

L, G, S_i, ($1 \leq i \leq c-1, i \in \mathbf{Z}$) and due to continuity, differentiability, convexity and monotonicity of function f(x), ($x \geq 0$).

In particular, it is well-known that a differentiable function of one variable is convex on an interval Ω if and only if its graph lies above all of its tangents: $f(x) \geq f(y) + f^{'}(y)\,(x-y)$, $x, y \in \Omega$ (see, e.g., Boyd and Vandenberghe [1, section 3.1.3]). □

Thus, instead of non-convex set $L \cap G$, we can consider a collection of right triangles: $\{ S_i \}$, so that search space of the problem (2): $(L \cap G) \cap \mathbf{Z}^2$ is identical to the union: $\cup (S_i \cap \mathbf{Z}^2)$, $1 \leq i \leq c-1, i \in \mathbf{Z}$.

Let us denote:

$$\mu_i := \min \{ (x^2 - a - by) \mid (x, y) \in S_i \cap \mathbf{Z}^2 \}, \quad (4)$$
$$1 \leq i \leq c-1, i \in \mathbf{Z}.$$

Theorem 1. $\mu = \min \{ \mu_i \mid 1 \leq i \leq c-1, i \in \mathbf{Z} \}$.

Proof. It follows from the above given definitions of μ, μ_i and Lemma 2.

Each problem (4) is Integer Quadratic Programming problem in the plane. According to Del Pia and Weismantel [4, Theorem 1.1], they can be solved in polynomial time.

Recall that polynomial-time algorithms are closed under union, composition, concatenation, intersection, complementation and some other operations: see, e.g., Hopcroft et al. [9, pp. 425−426].

That is why, due to Theorem 1, our original NP-complete problem (2) can be solved in polynomial time as well.

As a result, since due to the above algorithm, NP-complete problem can be solved in polynomial time, we can conclude that P = NP, since, as we mentioned above, if there is a polynomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP.

Since the original NP-complete problem is asking whether the corresponding minimum is zero or not, we can, finally, give the following algorithm (polynomial-time) for its solution:

Input: positive integers a, b, c.
Output: Zero_Or_Not.

Set Zero_Or_Not = "Not_Zero" .

for i = 1, ... , c − 1 **do**
  if min { $(x^2 - a - by)$ | $(x, y) \in S_i \cap \mathbf{Z}^2$ } = 0
  then Set Zero_Or_Not = "Zero"
  exit
  end
end

3. **Conclusion.** We reduced NP-complete problem to the polynomial-time algorithm, Thus, we can conclude that P = NP, since if there is a polynomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP.

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