Integer Optimization and P vs NP Problem

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Abstract

We transform NP-complete Problem to the polynomial-time algorithm which would mean that P = NP.

1 Introduction

Despite in general, Integer Programming is NP-hard or even incomputable (see, e.g., Hemmecke et al. [10]), for some subclasses of target functions and constraints it can be computed in time polynomial.

A fixed-dimensional polynomial minimization in integer variables, where the objective function is a convex polynomial and the convex feasible set is described by arbitrary polynomials can be solved in time polynomial(see, e.g Khachiyan and Porkolab [11]), see Lenstra [13] as well.

A fixed-dimensional polynomial minimization over the integer variables, where the objective function is a quasiconvex polynomial with integer coefficients and where the constraints are inequalities with quasiconvex polynomials of degree at most ≥ 2 with integer coefficients can be solved in time polynomial in the degrees and the binary encoding of the coefficients(see, e.g., Heinz [8], Hemmecke et al. [10], Lee [12]).

Minimizing a convex function over the integer points of a bounded convex set is polynomial in fixed dimension, according to Oertel et al. [15].

Del Pia and Weismantel [4] showed that Integer Quadratic Programming can be solved in polynomial time in the plane.

It was further generalized for cubic and homogeneous polynomials in Del Pia et al. [5].

We are going to transform well-known NP-complete problem to the polynomial-time integer minimization algorithm. It would mean, that P = NP, since if there is a polynomial-time algorithm for any NP-hard problem, then there are polynomial-time algorithms for all problems in NP (see Garey and Johnson [7], Manders and Adleman [14], Cormen et al. [2]).

Fortnow in [6] stated: "We call the very hardest NP problems (which include Partition Into Triangles, Clique, Hamiltonian Cycle and 3-Coloring)

"NP-complete", i.e. given an efficient algorithm for one of them, we can find efficient algorithm for all of them and in fact any problem in NP".

2 Polynomial-time Algorithm. Sliding Tangent

Lemma 1 (De Loera et al. [3], Hemmecke et al. [10], Del Pia et al. [5]).
   The problem of minimizing a degree-4 polynomial over the lattice points of a convex polygon is NP-hard.

Proof.   They use the NP-complete problem AN1 on page 249 of Garey and Johnson [7]. This problem states it is NP-complete to decide whether, given three positive integers a, b, c, there exists a positive integer $x < c$ such that x^2 is congruent with a modulo b. This problem is clearly equivalent to asking whether the minimum of the quartic polynomial function $(x^2 - a - by)^2$ over the lattice points of the rectangle:

$\{ (x,y) \mid 1 \le x \le c - 1, 1 - a \le by \le (c - 1)^2 - a \}$ is zero or not.   □

   According to Del Pia and Weismantel [4], minimization problem, given in the above proof of Lemma 1 is equivalent to the following problem:

$$\min \{ (x^2 - a - by) \text{ s.t.}$$
$$x^2 - a - by \ge 0, \tag{1}$$
$$1 \le x \le c - 1, 1 - a \le by \le (c - 1)^2 - a, \ x, y \in \mathbf{Z}\}.$$

   If $L := \{ (x, y) \in \mathbf{R}^2 \mid x^2 - a - by \ge 0, \ x \ge 0\}$,
      $G := \{ (x, y) \in \mathbf{R}^2 \mid 1 \le x \le c - 1, \ 1 - a \le by \le (c - 1)^2 - a \}$,

problem (1) can be rewritten as follows:

$$\mu := \min \{ (x^2 - a - by) \mid (L \cap G) \cap \mathbf{Z}^2 \}. \tag{2}$$

   Note that parabola: $by = bf(x) = x^2 - a, \ x \ge 0$ is a part of the border of set L and $bf(1) = 1 - a, bf(c - 1) = (c - 1)^2 - a$.
   Set L is not convex, as well as the set $L \cap G$ (see Boyd and Vandenberghe [1], Osborne [16]).
   The equation of the tangent to the parabola: $by = bf(x) = x^2 - a$, at the point i: $1 \le i \le c - 1, i \in \mathbf{Z}$, is given by: $by_i (x) = 2i (x - i) + i^2 - a$. The

segment of this tangent (hypotenuse), which is inside G and having one end on the horizontal line by $= 1 - a$, and another end on the vertical line $x = c - 1$, together with two other segments: on the horizontal line by $= 1 - a$ and on the vertical line $x = c - 1$, both segments intersected at the point (x_c, y_c): $x_c = c - 1$, by$_c = 1 - a$, cathetuses, form some right triangle $S_i \subset L \cap G$, $S_i :=$ $\{ (x, y) \in G \mid y \le y_i(x) \}$, $1 \le i \le c - 1$, $i \in \mathbf{Z}$.

Lemma 2. $(L \cap G) \cap \mathbf{Z}^2 = \cup(S_i \cap \mathbf{Z}^2)$, $1 \le i \le c - 1$, $i \in \mathbf{Z}$.

Proof.   It follows from the above given definitions and properties of sets L, G, S_i, $(1 \le i \le c - 1, i \in \mathbf{Z})$ and due to continuity, differentiability, convexity and monotonicity of function $f(x)$, $(x \ge 0)$.

In particular, it is well-known that a differentiable function of one variable is convex on an interval Ω if and only if its graph lies above all of its tangents: $f(x) \ge f(y) + f'(y)(x - y)$, $x, y \in \Omega$ (see, e.g., Boyd and Vandenberghe [1, section 3.1.3]).   □

Thus, instead of non-convex set $L \cap G$, we can consider a collection of right triangles: $\{ S_i \}$, so that search space of the problem (2): $(L \cap G) \cap \mathbf{Z}^2$ is identical to the union: $\cup (S_i \cap \mathbf{Z}^2)$, $1 \le i \le c - 1$, $i \in \mathbf{Z}$.

Let us denote:

$$\mu_i := \min \{ (x^2 - a - by) \mid (x, y) \in S_i \cap \mathbf{Z}^2 \}, \qquad (3)$$
$$1 \le i \le c - 1, i \in \mathbf{Z}.$$

Theorem 1. $\mu = \min \{ \mu_i \mid 1 \le i \le c - 1, i \in \mathbf{Z} \}$.

Proof.   It follows from the above given definitions of μ, μ_i and Lemma 2.

Each problem (3) is integer quadratic programming problem in the plane. According to Del Pia and Weismantel [4, Theorem 1.1], they can be solved in polynomial time.
Recall that polynomial-time algorithms are closed under union, composition, concatenation, intersection, complementation and some other operations: see, e.g., Hopcroft et al. [9, pp. 425−426].
That is why, due to Theorem 1, our original NP-complete problem (2) can be solved in polynomial time as well.

As a result, since due to the above algorithm, NP-complete problem can be solved in polynomial time, we can conclude that P = NP, since, as we mentioned above, if there is a polynomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP.

3 Conclusion

We reduced NP-complete problem to the polynomial-time algorithm, Thus, we can conclude that P = NP, since if there is a poly-nomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP.

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