Integer Optimization and P vs NP Problem

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Abstract. We transform NP-complete Problem to the polynomial-time algorithm which would mean that P = NP.

1. INTRODUCTION. Despite in general, Integer Programming is NP-hard or even incomputable (see, e.g., Hemmecke et al. [10]), for some subclasses of target functions and constraints it can be computed in time polynomial.

A fixed-dimensional polynomial minimization in integer variables, where the objective function is a convex polynomial and the convex feasible set is described by arbitrary polynomials can be solved in time polynomial(see, e.g ., Khachiyan and Porkolab [11]), see Lenstra [13] as well.

A fixed-dimensional polynomial minimization over the integer variables, where the objective function is a quasiconvex polynomial with integer coefficients and where the constraints are inequalities with quasiconvex polynomials of degree at most ≥ 2 with integer coefficients can be solved in time polynomial in the degrees and the binary encoding of the coefficients(see, e.g., Heinz [8], Hemmecke et al. [10], Lee [12]).

Minimizing a convex function over the integer points of a bounded convex set is polynomial in fixed dimension, according to Oertel et al. [15].

Del Pia and Weismantel [4] showed that Integer Quadratic Programming can be solved in polynomial time in the plane.

It was further generalized for cubic and homogeneous polynomials in Del Pia et al. [5].

We are going to transform well-known NP-complete problem to the polynomial-time integer minimization algorithm. It would mean, that P = NP, since if there is a polynomial-time algorithm for any NP-hard problem, then there are polynomial-time algorithms for all problems in NP (see Garey and Johnson [7], Manders and Adleman [14], Cormen et al. [2].).

Fortnow in [6] stated: "We call the very hardest NP problems (which include Partition Into Triangles, Clique, Hamiltonian Cycle and 3-Coloring) “NP-complete”, i.e. given an efficient algorithm for one of them, we can find efficient algorithm for all of them and in fact any problem in NP".

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2. POLYNOMIAL-TIME ALGORITHM. SLIDING TANGENT.

Lemma 1 (De Loera et al. [3], Hemmecke et al. [10], Del Pia et al. [5]).
The problem of minimizing a degree-4 polynomial over the lattice points of a convex polygon is NP-hard.

Proof. They use the NP-complete problem AN1 on page 249 of Garey and Johnson [7]. This problem states it is NP-complete to decide whether, given three positive integers a, b, c, there exists a positive integer $x < c$ such that x^2 is congruent with a modulo b. This problem is clearly equivalent to asking whether the minimum of the quartic polynomial function $(x^2 - a - by)^2$ over the lattice points of the rectangle:

$$\{ (x,y) \mid 1 \le x \le c - 1,\ 1 - a \le by \le (c-1)^2 - a \} \text{ is zero or not.} \qquad \square$$

According to Del Pia and Weismantel [4], minimization problem, given in the above proof of Lemma 1 is equivalent to the following problem:

$$\begin{aligned} \min \{ &(x^2 - a - by) \ \text{ s.t.} \\ &x^2 - a - by \ge 0, \\ &1 \le x \le c - 1,\ 1 - a \le by \le (c-1)^2 - a,\ x, y \in \mathbf{Z}\}. \end{aligned} \qquad (1)$$

Without loss of generality, let us consider the case, where in (1) a = b = 1, while c is an arbitrary sufficiently large positive fixed integer.
For the arbitrary fixed positive integers a and b it can be done similarly.

Thus, let us consider the following NP-complete problem:

$$\begin{aligned} \min \{ &(x^2 - 1 - y) \ \text{ s.t.} \\ &x^2 - 1 - y \ge 0, \\ &1 \le x \le c - 1,\ 0 \le y \le (c-1)^2 - 1,\ x, y \in \mathbf{Z}\}. \end{aligned} \qquad (2)$$

$$\begin{aligned} \text{If } \ L := &\{ (x, y) \in \mathbf{R}^2 \mid x^2 - 1 - y \ge 0 \}, \\ P := &\{ (x, y) \in \mathbf{R}^2 \mid 1 \le x \le c - 1,\ 0 \le y \le (c-1)^2 - 1 \}, \end{aligned}$$

problem (2) can be rewritten as follows:

$$\min \{ (x^2 - 1 - y) \mid (L \cap P) \cap \mathbf{Z}^2 \} \qquad (3)$$

Set L is not convex, as well as the set $L \cap P$ (see Boyd and Vandenberghe [1], Osborne [16] as well).

Let $1 \le i \le c - 1, i \in \mathbf{Z}$. The equation of the tangent to the parabola: $y = x^2 - 1$, at the point i is given by: $y_i(x) = 2i(x - i) + i^2 - 1$. The segment of this tangent(hypotenuse), which is inside P and having one end on X axis, and another end on the line $x = c - 1$, together with two other segments (on X axis and on the vertical line $x = c - 1$: cathetuses), form some right triangle $S_i \subset L \cap P$, $S_i := \{ (x, y) \in P \mid y \le y_i(x) \}, 1 \le i \le c - 1, i \in \mathbf{Z}$. Thus, instead of non-convex set $L \cap P$, we are going to consider a collection of right triangles: $\{ S_i \}$, so that search space of the problem (3):

$(L \cap P) \cap \mathbf{Z}^2 = \cup (S_i \cap \mathbf{Z}^2), 1 \le i \le c - 1, i \in \mathbf{Z}$. Let us denote:

$$\mu_i := \min \{ (x^2 - 1 - y) \mid (x, y) \in S_i \cap \mathbf{Z}^2 \}, 1 \le i \le c - 1, i \in \mathbf{Z}. \quad (4)$$

It is clear, that we have:

Theorem 1. *$\min \{ \mu_i \mid 1 \le i \le c - 1, i \in \mathbf{Z} \} = \mu = \min \{ (x^2 - 1 - y) \mid (L \cap P) \cap \mathbf{Z}^2 \}$.*

Each problem (4) is integer quadratic programming problem in the plane. According to Del Pia and Weismantel [4, Theorem 1.1], they can be solved in polynomial time.

Recall that polynomial-time algorithms are closed under union, composition, concatenation, intersection, complementation and some other operations: see, e.g., Hopcroft et al. [9, pp. 425−426].

That is why, due to Theorem 1, our original NP-complete problem (3) can be solved in polynomial time as well.

As we mentioned above, similarly, this algorithm can be developed for any fixed positive integers a and b as well.

As a result, since due to the above algorithm, NP-complete problem can be solved in polynomial time, we can conclude that P = NP, since, as we mentioned above, if there is a polynomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP.

3. CONCLUSION. We reduced NP-complete problem to the polynomial-time algorithm, Thus, we can conclude that P = NP, since if there is a polynomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP.

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