CAN EINSTEIN TENSOR BE GENERALIZED?

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Abstract. In this short paper I will write a possible generalizations of Einstein tensor and energy momentum tensor that will lead to generalizations of Einstein field equations.

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Contents

1. Einstein tensor

Einstein tensor [1] is basis of General Relativity, it has vacuum solutions equal to:

$$G^{\mu\nu} = 0 \tag{1.1}$$

Another property is that it is symmetric and it's covariant derivative is equal to zero from it follows that:

$$\nabla_\nu G^{\mu\nu} = 0 \tag{1.2}$$

$$G^{\mu\nu} = G^{\nu\mu} \tag{1.3}$$

It plays crucial role in Einstein field equations [2] as it is left side of field equation:

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.4}$$

Where tensor itself is equal to:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \tag{1.5}$$

So field equations are equal to :

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.6}$$

But in whole paper I will be using not contravariant form but covariant form of this tensor so $G_{\mu\nu}$. It will be same tensor but with covariant indexes, it will be equal to:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.7}$$

So field equation is same but with covariant indexes:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.8}$$

This is form of field equations I will use in whole paper.

2. Riemann tensor and generalized Einstein tensor

To build a generalized Einstein tensor I need to assume some kind of basis of deriving it. I will use Riemann tensor contractions are that basis, I want generalized tensor to have same contractions as Riemann tensor [3] [4]. It means that if i write Riemann tensor contractions they will be same as contractions of generalized Einstein tensor:

$$g^{\alpha\mu} R_{\alpha\mu\beta\nu} = 0 \tag{2.1}$$

$$g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R_{\mu\nu} \tag{2.2}$$

$$g^{\alpha\nu} R_{\alpha\mu\beta\nu} = -R_{\mu\beta} \tag{2.3}$$

$$g^{\mu\beta} R_{\alpha\mu\beta\nu} = -R_{\alpha\nu} \tag{2.4}$$

$$g^{\mu\nu} R_{\alpha\mu\beta\nu} = R_{\alpha\beta} \tag{2.5}$$

$$g^{\beta\nu} R_{\alpha\mu\beta\nu} = 0 \tag{2.6}$$

So I can write down now same contractions but for generalized Einstein tensor $G_{\alpha\mu\beta\nu}$:

$$g^{\alpha\mu} G_{\alpha\mu\beta\nu} = 0 \tag{2.7}$$

$$g^{\alpha\beta} G_{\alpha\mu\beta\nu} = G_{\mu\nu} \tag{2.8}$$

$$g^{\alpha\nu} G_{\alpha\mu\beta\nu} = -G_{\mu\beta} \tag{2.9}$$

$$g^{\mu\beta} G_{\alpha\mu\beta\nu} = -G_{\alpha\nu} \tag{2.10}$$

$$g^{\mu\nu} G_{\alpha\mu\beta\nu} = G_{\alpha\beta} \tag{2.11}$$

$$g^{\beta\nu} G_{\alpha\mu\beta\nu} = 0 \tag{2.12}$$

From it comes another part of equations that is generalized energy momentum tensor [5], that will have same contraction properties as Riemann tensor and generalized Einstein tensor to follow a field equation:

$$g^{\alpha\mu} T_{\alpha\mu\beta\nu} = 0 \tag{2.13}$$

$$g^{\alpha\beta} T_{\alpha\mu\beta\nu} = T_{\mu\nu} \tag{2.14}$$

$$g^{\alpha\nu} T_{\alpha\mu\beta\nu} = -T_{\mu\beta} \tag{2.15}$$

$$g^{\mu\beta} T_{\alpha\mu\beta\nu} = -T_{\alpha\nu} \tag{2.16}$$

$$g^{\mu\nu} T_{\alpha\mu\beta\nu} = T_{\alpha\beta} \tag{2.17}$$

$$g^{\beta\nu} T_{\alpha\mu\beta\nu} = 0 \tag{2.18}$$

So from it comes that generalized Einstein tensor reduces either to plus-minus Einstein tensor or zero and generalized energy momentum tensor have to obey same rule to make it consistent with field equations.

3. Generalized Einstein tensor

I will first write generalized Einstein tensor and generalized energy momentum tensor, then will show that they indeed follow contractions properties. So those tensors are equal to:

$$G_{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) \tag{3.1}$$

$$\begin{aligned} T_{\alpha\mu\beta\nu} = & \frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) \\ & -\frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) \end{aligned} \tag{3.2}$$

It's easy to check that all those tensors contractions agree with previous assumptions, so it contractions lead to field equation or zero:

$$g^{\alpha\mu}G_{\alpha\mu\beta\nu} = 0 \tag{3.3}$$

$$g^{\alpha\beta}G_{\alpha\mu\beta\nu} = G_{\mu\nu} \tag{3.4}$$

$$g^{\alpha\nu}G_{\alpha\mu\beta\nu} = -G_{\mu\beta} \tag{3.5}$$

$$g^{\mu\beta}G_{\alpha\mu\beta\nu} = -G_{\alpha\nu} \tag{3.6}$$

$$g^{\mu\nu}G_{\alpha\mu\beta\nu} = G_{\alpha\beta} \tag{3.7}$$

$$g^{\beta\nu}G_{\alpha\mu\beta\nu} = 0 \tag{3.8}$$

$$g^{\alpha\mu}T_{\alpha\mu\beta\nu} = 0 \tag{3.9}$$

$$g^{\alpha\beta}T_{\alpha\mu\beta\nu} = T_{\mu\nu} \tag{3.10}$$

$$g^{\alpha\nu}T_{\alpha\mu\beta\nu} = -T_{\mu\beta} \tag{3.11}$$

$$g^{\mu\beta}T_{\alpha\mu\beta\nu} = -T_{\alpha\nu} \tag{3.12}$$

$$g^{\mu\nu}T_{\alpha\mu\beta\nu} = T_{\alpha\beta} \tag{3.13}$$

$$g^{\beta\nu}T_{\alpha\mu\beta\nu} = 0 \tag{3.14}$$

It means that if I do pair this two tensors and contract them I will get zero or Einstein field equation. That property is key to whole idea of generalizing those equations they will lead back to Einstein equations or will just give zero so contain no information at all. Those two tensors are simplest tensors that follow Riemann tensor contraction properties. Where generalized energy momentum tensor is created out of energy momentum tensor and metric tensor.

4. Conservation

It easy to derive that covariant derivative of extended Einstein tensor is equal to zero for all indexes. It means that:

$$g^{\alpha\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.1}$$

$$g^{\mu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.2}$$

$$g^{\beta\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.3}$$

$$g^{\nu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.4}$$

Where i did use covariant derivative with upper index index raising, covariant derivative of metric tensor is equal to zero so I can treat metric tensor as a constant, from fact that tensor reduces to Einstein tensor so if I multiply one side of equation by metric tensor I will still get valid result and from fact that after using metric tensor to contract generalized Einstein tensor:

$$g^{\mu\phi}\nabla_{\phi}G_{\mu\nu} = 0 \tag{4.5}$$

$$g^{\mu\phi}\nabla_{\phi}g^{\alpha\beta}G_{\alpha\mu\beta\nu} = g^{\mu\phi}\nabla_{\phi}G_{\mu\nu} = 0 \tag{4.6}$$

$$g_{\alpha\beta}g^{\alpha\beta}g^{\mu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.7}$$

$$4g^{\mu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.8}$$

$$g^{\mu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.9}$$

$$g^{\nu\phi}\nabla_{\phi}G_{\mu\nu} = 0 \tag{4.10}$$

$$g^{\nu\phi}\nabla_{\phi}g^{\alpha\beta}G_{\alpha\mu\beta\nu} = g^{\nu\phi}\nabla_{\phi}G_{\mu\nu} = 0 \tag{4.11}$$

$$g_{\alpha\beta}g^{\alpha\beta}g^{\nu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.12}$$

$$4g^{\nu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.13}$$

$$g^{\nu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.14}$$

$$g^{\alpha\phi}\nabla_{\phi}G_{\alpha\beta} = 0 \tag{4.15}$$

$$g^{\alpha\phi}\nabla_{\phi}g^{\mu\nu}G_{\alpha\mu\beta\nu} = g^{\alpha\phi}\nabla_{\phi}G_{\alpha\beta} = 0 \tag{4.16}$$

$$g_{\alpha\beta}g^{\alpha\beta}g^{\alpha\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.17}$$

$$4g^{\alpha\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.18}$$

$$g^{\alpha\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.19}$$

$$g^{\beta\phi}\nabla_{\phi}G_{\alpha\beta} = 0 \tag{4.20}$$

$$g^{\beta\phi}\nabla_{\phi}g^{\mu\nu}G_{\alpha\mu\beta\nu} = g^{\beta\phi}\nabla_{\phi}G_{\alpha\beta} = 0 \tag{4.21}$$

$$g_{\alpha\beta}g^{\alpha\beta}g^{\beta\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.22}$$

$$4g^{\beta\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.23}$$

$$g^{\beta\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.24}$$

For generalized energy momentum tensor case is even simpler it's build form energy momentum tensor and metric tensors both of them will vanish when taking the covariant derivative.

5. Field equation

Generalized Einstein field equation takes form equal to:

$$G_{\alpha\mu\beta\nu} = \kappa T_{\alpha\mu\beta\nu} \tag{5.1}$$

That can be expanded to see all the components of field equation as:

$$\begin{aligned} &R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) \\ &= \kappa\left[\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right] \end{aligned} \tag{5.2}$$

This equations contractions lead to zero or Einstein field equations. Equation itself is very complex, both generalized energy momentum tensor and generalized Einstein tensor part without Riemann tensor follow Riemann tensor vanishing properties. It means that if I take two first or two last indexes to be equal both side of equation will vanish. Another property is that when Riemann tensor components that are not independent change sign of independent components so does generalized energy momentum tensor and part with Ricci tensor from generalized Einstein tensor. So in general it follows Riemann tensor properties that makes field equation consistent. I can write those properties:

$$T_{\alpha\alpha\beta\nu} = 0 \tag{5.3}$$

$$T_{\alpha\mu\beta\beta} = 0 \tag{5.4}$$

$$T_{\alpha\mu\beta\nu} = -T_{\mu\alpha\beta\nu} \tag{5.5}$$

$$T_{\alpha\mu\beta\nu} = -T_{\alpha\mu\nu\beta} \tag{5.6}$$

$$T_{\alpha\mu\beta\nu} + T_{\alpha\nu\mu\beta} + T_{\alpha\beta\nu\mu} = 0 \tag{5.7}$$

All of those are easy to prove I will write last one:

$$\begin{aligned} &\left[\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right] \\ &+\left[\frac{1}{2}\left(T_{\alpha\mu}g_{\nu\beta} + T_{\nu\beta}g_{\alpha\mu}\right) - \frac{1}{2}\left(T_{\mu\nu}g_{\alpha\beta} + T_{\alpha\beta}g_{\mu\nu}\right) - \frac{1}{6}T\left(g_{\alpha\mu}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta}\right)\right] \\ &+\left[\frac{1}{2}\left(T_{\alpha\nu}g_{\beta\mu} + T_{\beta\mu}g_{\alpha\nu}\right) - \frac{1}{2}\left(T_{\beta\nu}g_{\alpha\mu} + T_{\alpha\mu}g_{\beta\nu}\right) - \frac{1}{6}T\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu}\right)\right] = 0 \end{aligned} \tag{5.8}$$

Same applies for part with Ricci Scalar:

$$\frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) + \frac{1}{6}R\left(g_{\alpha\mu}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{6}R\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu}\right) = 0 \tag{5.9}$$

It proves that field equation follows all Riemann tensor properties.

6. Trace reversed field equation

Field equations can lead to equation that expresses Riemann tensor in terms of energy momentum tensor and metric tensor combinations. It means that all spacetime curvature is expressed in terms of energy momentum and metric tensors that will lead to need of non vanishing energy momentum tensor. To arrive at trace reversed field equations I need to start with Einstein field equations, then contract them with metric tensor to arrive at:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.1}$$

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu} \tag{6.2}$$

$$R - \frac{D}{2} R = \kappa T \quad D = 4 \tag{6.3}$$

$$R - 2R = \kappa T \tag{6.4}$$

$$-R = \kappa T \tag{6.5}$$

Now I can replace Ricci scalar with trace of energy momentum tensor in field equations so I will arrive at Riemann tensor equal to:

$$\begin{aligned} &R_{\alpha\mu\beta\nu} + \frac{1}{6}\kappa T \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) \\ = \kappa &\left[\frac{1}{2} \left(T_{\alpha\beta} g_{\mu\nu} + T_{\mu\nu} g_{\alpha\beta} \right) - \frac{1}{2} \left(T_{\mu\beta} g_{\alpha\nu} + T_{\alpha\nu} g_{\mu\beta} \right) - \frac{1}{6} T \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) \right] \end{aligned} \tag{6.6}$$

$$R_{\alpha\mu\beta\nu} = \kappa \left[\frac{1}{2} \left(T_{\alpha\beta} g_{\mu\nu} + T_{\mu\nu} g_{\alpha\beta} \right) - \frac{1}{2} \left(T_{\mu\beta} g_{\alpha\nu} + T_{\alpha\nu} g_{\mu\beta} \right) - \frac{1}{3} T \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) \right] \tag{6.7}$$

Those trace reversed field equations if contracted will lead to trace reversed Einstein field equations or to zero. That is needed in order to make this model consistent with itself. Trace reversed field equation point to important property of field equations, if I set energy momentum tensor to zero so it will vanish at any point of space those equations will lead to flat spacetime. It means that source of curvature of spacetime is directly a matter field there are no vacuum solutions to this equation.

7. Non vanishing energy momentum tensor

Energy momentum tensor [6] is source of matter field in general relativity, but simpler way is to solve vacuum field equations, in my model there are not vacuum solutions. It means that matter field does not vanish at any point of space or otherwise it leads to flat spacetime. It means that energy momentum tensor has to be expressed as non vanishing at any point of space. It means that source of deviation from flat spacetime is matter itself. Energy momentum tensor has to be conserved locally but not in general sense. It means that it will vanish with respect to covariant derivative:

$$\nabla_\nu T^{\mu\nu} = 0 \tag{7.1}$$

But on the other hand if that tensor depends on some coordinates system $\mathbf{x}$ it's non zero for whole manifold or saying it another way it's defined to be non-zero for whole manifold that gives condition there is either flat spacetime or there is matter in whole space. It means that source of gravity field of a point like matter has to be matter field that extend that point like matter into whole space. This gives condition that matter field is responsible for all gravity effects in any point of space. If I have matter at some point $\mathbf{x}_A$ there is condition that whole space is covered with matter field so energy momentum tensor does not vanish at any given point:

$$T^{\mu\nu}(\mathbf{x}_A) \neq 0 \to T^{\mu\nu}(\mathbf{x}) \neq 0 \tag{7.2}$$

It gives another view on gravity field itself, it's caused by a matter field so matter field has to threat as one object extended in whole space. Simplest model of that kind of matter field is a source matter field. There is source of matter field that has maximum value of field (where I assume only one object with energy) then it spreads in whole field with energy equal to it's gravitation effects at given point. It means that matter field acts as mediation of gravity effects, space is not empty even if there is seemingly one particle present. This could in principle explain dark matter. But this point like approach will generate singularity at radius equal to zero, so another approach is that rest mass in not concentrated in point but rather spread out in given volume, this approach is more realistic and by itself does not lead to singularity at radius equal to zero as long as mass vanish there. Both approaches or any given approach will have to fulfill two rules, first off it has to generates rest mass of particle for given volume of space and second it has to generate gravity effects far away from rest mass.

8. Dimensions

In general this equation works with four dimensional spacetime as there there is equal amount of unknowns on both side of equation. In higher dimensions this will have unequal amount of unknowns that could mean that it's wrong in any other dimensions that four or that there is some symmetry for Riemann tensor when going into higher dimensional spacetime. Now I can write a same equation but for more than four dimensions, it will be change by constants where I do denote number of dimensions as D:

$$\begin{gathered} R_{\alpha\mu\beta\nu} - \frac{1}{2(D-1)} R \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) \\ = \frac{2\kappa}{D} \left[\left(T_{\alpha\beta} g_{\mu\nu} + T_{\mu\nu} g_{\alpha\beta} \right) - \left(T_{\mu\beta} g_{\alpha\nu} + T_{\alpha\nu} g_{\mu\beta} \right) \right] - \frac{\kappa}{2(D-1)} T \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) \end{gathered} \tag{8.1}$$

Now to arrive at trace reversed field equations I need to go back to Ricci scalar relation with trace of energy momentum tensor, from it I can figure out general equation that will give me:

$$\left(1 - \frac{D}{2}\right) R = \kappa T \tag{8.2}$$

$$R = \kappa \frac{1}{\left(1 - \frac{D}{2}\right)} T \tag{8.3}$$

Now what is left is to plug into field equations this equality and I will get:

$$\begin{gathered} R_{\alpha\mu\beta\nu} = \frac{2\kappa}{D} \left[\left(T_{\alpha\beta} g_{\mu\nu} + T_{\mu\nu} g_{\alpha\beta} \right) - \left(T_{\mu\beta} g_{\alpha\nu} + T_{\alpha\nu} g_{\mu\beta} \right) \right] \\ + \frac{\kappa}{2(D-1)} \left[\frac{1}{\left(1 - \frac{D}{2}\right)} T \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) - T \left(g_{\alpha\beta} g_{\mu\nu} - g_{\beta\mu} g_{\alpha\nu} \right) \right] \end{gathered} \tag{8.4}$$

This equation will correctly reduce to trace reversed field equation, for higher dimensions that four as with dimensions four. Key question is that this dependence of amount of unknowns between Riemann tensor part and energy momentum tensor part means that equation works only in four dimensions or it will work in higher dimensions , when there is not equal number of unknowns in equation it means that it's unsolvable in general sense. But if i threat that there are only unknowns on energy momentum part for dimension D it mean that Riemann tensor will have to obey additional symmetry rules that could break it. So in principle i could chose that independent components of only energy momentum part but still will have to solve more equations than there is independent components. So it leaves as an open question could this model work in higher dimensions but most probably no.

9. Meaning of field equations

Field equations are complex equations. Simplest interpretation has to do with Riemann tensor interpretation. I will focus on four dimensional case for simplicity and not writing all the constants that depend on number of dimensions. Let me start by trace reversed field equations. Geodesic deviation [7] is meaning behind Riemann tensor, so I can re-write field equation as, where i start by rising one index:

$$g^{\gamma\alpha}R_{\alpha\mu\beta\nu}$$
$$= \kappa g^{\gamma\alpha}\left[\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu}+T_{\mu\nu}g_{\alpha\beta}\right)-\frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu}+T_{\alpha\nu}g_{\mu\beta}\right)-\frac{1}{3}T\left(g_{\alpha\beta}g_{\mu\nu}-g_{\beta\mu}g_{\alpha\nu}\right)\right] \tag{9.1}$$

$$R^{\gamma}{}_{\mu\beta\nu} = \kappa\left[\frac{1}{2}\left(T^{\gamma}_{\beta}g_{\mu\nu}+\delta^{\gamma}_{\beta}T_{\mu\nu}\right)-\frac{1}{2}\left(\delta^{\gamma}_{\nu}T_{\mu\beta}+T^{\gamma}_{\nu}g_{\mu\beta}\right)-\frac{1}{3}T\left(\delta^{\gamma}_{\beta}g_{\mu\nu}-\delta^{\gamma}_{\nu}g_{\beta\mu}\right)\right] \tag{9.2}$$

Now I can write geodesic deviation equation for this field equation, that will use Riemann tensor:

$$\frac{D^2Y^{\gamma}}{d\lambda^2} = R^{\gamma}{}_{\mu\beta\nu}u^{\mu}u^{\nu}Y^{\beta}$$
$$= \kappa\left[\frac{1}{2}\left(T^{\gamma}_{\beta}g_{\mu\nu}+\delta^{\gamma}_{\beta}T_{\mu\nu}\right)-\frac{1}{2}\left(\delta^{\gamma}_{\nu}T_{\mu\beta}+T^{\gamma}_{\nu}g_{\mu\beta}\right)-\frac{1}{3}T\left(\delta^{\gamma}_{\beta}g_{\mu\nu}-\delta^{\gamma}_{\nu}g_{\beta\mu}\right)\right]u^{\mu}u^{\nu}Y^{\beta} \tag{9.3}$$

Where Y^{β} is geodesic separation vector, u^{μ}, u^{ν} is four velocity vector if I chose affine parameter to be proper time, and D^2 denotes covariant acceleration of separation vector Y^{β}. So from this equation come simplest interpretation of field equation, it says at what rate geodesic accelerate towards or away from each other. Those create tidal forces or just stretch objects in space. Again this equation will be equal to zero if there is no matter field present in whole space.

References

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