CAN EINSTEIN TENSOR BE GENERALIZED?

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Abstract. In this short paper I will write a possible generalizations of Einstein tensor and energy momentum tensor that will lead to generalizations of Einstein field equations.

Date: April 14, 2024.

Contents

1. Einstein tensor

Einstein tensor [1] is basis of General Relativity, it has vacuum solutions equal to:

$$G^{\mu\nu} = 0 \tag{1.1}$$

Another property is that it is symmetric and it's covariant derivative is equal to zero from it follows that:

$$\nabla_\nu G^{\mu\nu} = 0 \tag{1.2}$$

$$G^{\mu\nu} = G^{\nu\mu} \tag{1.3}$$

It plays crucial role in Einstein field equations [2] as it is left side of field equation:

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.4}$$

Where tensor itself is equal to:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} \tag{1.5}$$

So field equations are equal to :

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.6}$$

But in whole paper I will be using not contravariant form but covariant form of this tensor so $G_{\mu\nu}$. It will be same tensor but with covariant indexes, it will be equal to:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{1.7}$$

So field equation is same but with covariant indexes:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.8}$$

This is form of field equations I will use in whole paper.

2. Riemann tensor and generalized Einstein tensor

To build a generalized Einstein tensor I need to assume some kind of basis of deriving it. I will use Riemann tensor contractions are that basis, I want generalized tensor to have same contractions as Riemann tensor [3] [4]. It means that if i write Riemann tensor contractions they will be same as contractions of generalized Einstein tensor:

$$g^{\alpha\mu}R_{\alpha\mu\beta\nu} = 0 \tag{2.1}$$

$$g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R_{\mu\nu} \tag{2.2}$$

$$g^{\alpha\nu}R_{\alpha\mu\beta\nu} = -R_{\mu\beta} \tag{2.3}$$

$$g^{\mu\beta}R_{\alpha\mu\beta\nu} = -R_{\alpha\nu} \tag{2.4}$$

$$g^{\mu\nu}R_{\alpha\mu\beta\nu} = R_{\alpha\beta} \tag{2.5}$$

$$g^{\beta\nu}R_{\alpha\mu\beta\nu} = 0 \tag{2.6}$$

So I can write down now same contractions but for generalized Einstein tensor $G_{\alpha\mu\beta\nu}$:

$$g^{\alpha\mu}G_{\alpha\mu\beta\nu} = 0 \tag{2.7}$$

$$g^{\alpha\beta}G_{\alpha\mu\beta\nu} = G_{\mu\nu} \tag{2.8}$$

$$g^{\alpha\nu}G_{\alpha\mu\beta\nu} = -G_{\mu\beta} \tag{2.9}$$

$$g^{\mu\beta}G_{\alpha\mu\beta\nu} = -G_{\alpha\nu} \tag{2.10}$$

$$g^{\mu\nu}G_{\alpha\mu\beta\nu} = G_{\alpha\beta} \tag{2.11}$$

$$g^{\beta\nu}G_{\alpha\mu\beta\nu} = 0 \tag{2.12}$$

From it comes another part of equations that is generalized energy momentum tensor [5], that will have same contraction properties as Riemann tensor and generalized Einstein tensor to follow a field equation:

$$g^{\alpha\mu}T_{\alpha\mu\beta\nu} = 0 \tag{2.13}$$

$$g^{\alpha\beta}T_{\alpha\mu\beta\nu} = T_{\mu\nu} \tag{2.14}$$

$$g^{\alpha\nu}T_{\alpha\mu\beta\nu} = -T_{\mu\beta} \tag{2.15}$$

$$g^{\mu\beta}T_{\alpha\mu\beta\nu} = -T_{\alpha\nu} \tag{2.16}$$

$$g^{\mu\nu}T_{\alpha\mu\beta\nu} = T_{\alpha\beta} \tag{2.17}$$

$$g^{\beta\nu}T_{\alpha\mu\beta\nu} = 0 \tag{2.18}$$

So from it comes that generalized Einstein tensor reduces either to plus-minus Einstein tensor or zero and generalized energy momentum tensor have to obey same rule to make it consistent with field equations.

3. Generalized Einstein tensor

I will first write generalized Einstein tensor and generalized energy momentum tensor, then will show that they indeed follow contractions properties. So those tensors are equal to:

$$G_{\alpha\mu\beta\nu} = 2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right) \tag{3.1}$$

$$T_{\alpha\mu\beta\nu} = \frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) \tag{3.2}$$

$$g^{\alpha\mu}\left(2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right)\right) = 0 \tag{3.3}$$

$$g^{\alpha\beta}\left(2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right)\right) = G_{\mu\nu} \tag{3.4}$$

$$g^{\alpha\nu}\left(2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right)\right) = -G_{\mu\beta} \tag{3.5}$$

$$g^{\mu\beta}\left(2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right)\right) = -G_{\alpha\nu} \tag{3.6}$$

$$g^{\mu\nu}\left(2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right)\right) = G_{\alpha\beta} \tag{3.7}$$

$$g^{\beta\nu}\left(2R_{\alpha\mu\beta\nu} - \frac{1}{2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right) + \frac{1}{2}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu}\right)\right) = 0 \tag{3.8}$$

$$g^{\alpha\mu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{3.9}$$

$$g^{\alpha\beta}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = T_{\mu\nu} \tag{3.10}$$

$$g^{\alpha\nu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = -T_{\mu\beta} \tag{3.11}$$

$$g^{\mu\beta}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = -T_{\alpha\nu} \tag{3.12}$$

$$g^{\mu\nu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = T_{\alpha\beta} \tag{3.13}$$

$$g^{\beta\nu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{3.14}$$

4. Summary

In this short paper I showed possible generalization of Einstein tensor. This leads to generalized energy momentum tensor, that combined create a new field equation:

$$G_{\alpha\mu\beta\nu} = \kappa T_{\alpha\mu\beta\nu} \tag{4.1}$$

Contractions of this field equation lead to zero or plus-minus Einstein tensor. That gives new equation for space-time curvature and new vacuum equations that will be equal to:

$$G_{\alpha\mu\beta\nu} = 0 \tag{4.2}$$

Problem with this equation is that is really hard to solve, as its a four rank tensor. For example field equation will take form for simplest case of vacuum:

$$2R_{\alpha\beta\alpha\beta} - \frac{1}{2}\left(R_{\alpha\alpha}g_{\beta\beta} + R_{\beta\beta}g_{\alpha\alpha}\right) + \frac{1}{2}\left(R_{\alpha\beta}g_{\alpha\beta} + R_{\alpha\beta}g_{\beta\alpha}\right) = 0 \tag{4.3}$$

From fact that independent components for Riemann tensor in case of spherical symmetric space-time are only six of them [6] and there are no cross terms for Ricci tensor I will get:

$$2R_{\alpha\beta\alpha\beta} - \frac{1}{2}\left(R_{\alpha\alpha}g_{\beta\beta} + R_{\beta\beta}g_{\alpha\alpha}\right) = 0 \tag{4.4}$$

Where I can write independent components:

$$2R_{0101} - \frac{1}{2}\left(R_{00}g_{11} + R_{11}g_{00}\right) = 0 \tag{4.5}$$

$$2R_{0202} - \frac{1}{2}\left(R_{00}g_{22} + R_{22}g_{00}\right) = 0 \tag{4.6}$$

$$2R_{0303} - \frac{1}{2}\left(R_{00}g_{33} + R_{33}g_{00}\right) = 0 \tag{4.7}$$

$$2R_{1212} - \frac{1}{2}\left(R_{11}g_{22} + R_{22}g_{11}\right) = 0 \tag{4.8}$$

$$2R_{1313} - \frac{1}{2}\left(R_{11}g_{33} + R_{33}g_{11}\right) = 0 \tag{4.9}$$

$$2R_{2323} - \frac{1}{2}\left(R_{22}g_{33} + R_{33}g_{22}\right) = 0 \tag{4.10}$$

From it follows clearly that vacuum solutions have non-vanishing Ricci tensor, even in simplest case.

References

[1] https://mathworld.wolfram.com/EinsteinTensor.html

[2] https://mathworld.wolfram.com/EinsteinFieldEquations.html

[3] https://mathworld.wolfram.com/RicciCurvatureTensor.html

[4] https://mathworld.wolfram.com/RiemannTensor.html

[5] https://www.astro.gla.ac.uk/users/martin/teaching/gr1/gr1_sec08.pdf

[6] http://web.mit.edu/klmitch/classes/8.033/Schwarzschild.pdf

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