Formulae of vector analysis

A. V. Serghienko

Abstract

We derive the formulae for the sine and the cosine of the sum, not using the notions of scalar and vector products, and using only the definitions of the sine and the cosine.

We derive the formulae for the gradient operator, the divergence and the Laplace operator in different orthogonal coordinate systems, not using any additional constructions like Lamé coefficients, and using only the definitions of the sine and the cosine.

Let $\vec{u}_\varphi$ is a unit vector on the plane, in which the unit vector $\vec{e}_x$ passes at the rotation by the angle φ counterclockwise. Then

$$\vec{e}_x = \vec{u}_0,\ \ \vec{e}_y = \vec{u}_{\frac{\pi}{2}}.$$

Under the definitions of sine and cosine the relationships take place:

$$\vec{u}_\alpha = \cos\alpha\vec{e}_x + \sin\alpha\vec{e}_y, \tag{1}$$

$$\vec{u}_{\frac{\pi}{2}+\alpha} = \cos\alpha\vec{e}_y - \sin\alpha\vec{e}_x, \tag{2}$$

$$\vec{u}_{\alpha+\beta} = \cos\beta\vec{u}_\alpha + \sin\beta\vec{u}_{\frac{\pi}{2}+\alpha}, \tag{3}$$

$$\vec{u}_{\alpha+\beta} = \cos\left(\alpha+\beta\right)\vec{e}_x + \sin\left(\alpha+\beta\right)\vec{e}_y. \tag{4}$$

Substituting (1) and (2) in (3) and applying (4), we obtain:

$$\begin{aligned} \cos\left(\alpha+\beta\right) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha,\\ \sin\left(\alpha+\beta\right) &= \cos\beta\sin\alpha + \sin\beta\cos\alpha. \end{aligned} \tag{5}$$

Write the relationships for unit vectors in different orthogonal coordinate systems:

$$\begin{aligned} \vec{e}_x &= \cos\varphi\vec{e}_\rho - \sin\varphi\vec{e}_\varphi, &\quad \vec{e}_z &= \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta,\\ \vec{e}_y &= \cos\varphi\vec{e}_\varphi + \sin\varphi\vec{e}_\rho, &\quad \vec{e}_\rho &= \cos\theta\vec{e}_\theta + \sin\theta\vec{e}_r; \end{aligned} \tag{6}$$

$$\begin{aligned} \vec{e}_\rho &= \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y, &\quad \vec{e}_r &= \cos\theta\vec{e}_z + \sin\theta\vec{e}_\rho,\\ \vec{e}_\varphi &= \cos\varphi\vec{e}_y - \sin\varphi\vec{e}_x, &\quad \vec{e}_\theta &= \cos\theta\vec{e}_\rho - \sin\theta\vec{e}_z. \end{aligned} \tag{7}$$

For differential operators (at the action on a continuous function) the analogous relationships take place:

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\varphi\frac{\partial}{\partial\rho} - \sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}, \quad & \frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial y} &= \cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\rho}, \quad & \frac{\partial}{\partial \rho} &= \cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}; \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}, \quad & \frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial z} + \sin\theta\frac{\partial}{\partial \rho}, \\ \frac{1}{\rho}\frac{\partial}{\partial\varphi} &= \cos\varphi\frac{\partial}{\partial y} - \sin\varphi\frac{\partial}{\partial x}, \quad & \frac{1}{r}\frac{\partial}{\partial\theta} &= \cos\theta\frac{\partial}{\partial \rho} - \sin\theta\frac{\partial}{\partial z}. \end{aligned} \tag{9}$$

Expand a vector $\vec{a}$ in different orthogonal coordinates:

$$\vec{a} = a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z, \tag{10}$$

$$\vec{a} = a_\rho\vec{e}_\rho + a_\varphi\vec{e}_\varphi + a_z\vec{e}_z, \tag{11}$$

$$\vec{a} = a_r\vec{e}_r + a_\theta\vec{e}_\theta + a_\varphi\vec{e}_\varphi. \tag{12}$$

Substituting in (11) the expressions for $\vec{e}_\rho$, $\vec{e}_\varphi$ and comparing with (10), we obtain:

$$\begin{aligned} a_x &= a_\rho\cos\varphi - a_\varphi\sin\varphi, \\ a_y &= a_\rho\sin\varphi + a_\varphi\cos\varphi. \end{aligned} \tag{13}$$

Substituting in (12) the expressions for $\vec{e}_r$, $\vec{e}_\theta$ and comparing with (11), we obtain:

$$\begin{aligned} a_z &= a_r\cos\theta - a_\theta\sin\theta, \\ a_\rho &= a_r\sin\theta + a_\theta\cos\theta. \end{aligned} \tag{14}$$

The gradient operator in Cartesian coordinates has the form

$$\vec{\nabla} = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}. \tag{15}$$

Substituting here the expressions for $\vec{e}_x$, $\vec{e}_y$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, we obtain the gradient operator in cylindrical coordinates:

$$\vec{\nabla} = \vec{e}_\rho\frac{\partial}{\partial\rho} + \vec{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \vec{e}_z\frac{\partial}{\partial z}. \tag{16}$$

Substituting here the expressions for $\vec{e}_z$, $\vec{e}_\rho$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial\rho}$, $\rho = r\sin\theta$, we obtain the gradient operator in spherical coordinates:

$$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{17}$$

The divergence of the vector $\vec{a}$ in Cartesian coordinates has the form

$$\mathrm{div}\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}. \tag{18}$$

Substituting here the expressions for $a_x,\ a_y,\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y}$, we obtain for the divergence in cylindrical coordinates:

$$\mathrm{div}\vec{a} = \frac{\partial a_\rho}{\partial \rho} + \frac{a_\rho}{\rho} + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}. \tag{19}$$

Substituting here the expressions for $a_z,\ a_\rho,\ \frac{\partial}{\partial z},\ \frac{\partial}{\partial \rho},\ \rho = r\sin\theta$, we obtain for the divergence in spherical coordinates:

$$\mathrm{div}\vec{a} = \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\cot\theta}{r}a_\theta + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi}. \tag{20}$$

By definition

$$\Delta u = \mathrm{div}\vec{a},\ \text{ where } \vec{a} = \vec{\nabla} u. \tag{21}$$

Using the formulae (15)–(21), we obtain for the Laplace operator in different orthogonal coordinate systems:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{22}$$

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \tag{23}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}. \tag{24}$$