Hidden nonlinearity of weak field sound wave in (2+1)-dimensional empty space-time

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We propose that the weak field sound wave derived from Newton's second law of gravitation in (2+1)-dimensional empty space-time, a linear equation, could have hidden nonlinearity.

Keywords: *nonlinearity, weak field, sound wave, Newton's second law, (2+1)-dimensional empty space-time.*

I. INTRODUCTION

It is commonly believed that there exists no nonlinearity in Newton's theory of gravitation (Newtonian field equation, Newton's second law of gravitation)[1–6]. It is because explicitly Newton's theory of gravitation is written in a linear equation. Related to the general theory of relativity, Newton's theory of gravitation is the weak-field limit of Einstein's non-linear theory of gravitation. On the other side, sound is a branch of mechanics (i.e. fluid mechanics), and so it is to be understood in terms of Newton's law[7], especially Newton's second law. If we interpret the acceleration in Newton's second law as the gravitational field then we obtain Newton's second law of gravitation, a linear equation. So, how could there exist a nonlinearity in a sound wave?

Inspired by the works of Ranada[8,9], we assume that the curvature tensor (the set of the solutions of Einstein field equation) in an empty space-time consists of subset fields, scalar fields. We call these subset fields subset curvature. In the case of empty space-time or weak field, the curvature tensor satisfies a linear subset curvature equation, but subset curvature satisfy linear and non-linear subset curvature equations. It means that, in the case of empty space-time or weak field, a non-linear subset curvature theory reduces to a linear subset curvature theory, in this case, it can be Newton's linear theory of gravitation.

Subset curvature are locally equal to the curvature tensor i.e. the curvature tensor can be obtained by patching together subset curvature (except in a zero-measure set) but globally different. The difference between the subset curvature and the curvature tensor in an empty space-time or weak field is global instead of local since the subset curvature obey the topological quantum condition but the curvature tensor does not.

In this article, we propose that the weak field sound wave derived from Newton's second law of gravitation in (2+1)-dimensional empty space-time could have hidden nonlinearity. What we mean by the weak field in sound wave is the weak pressure. This nonlinearity could exist because Newton's theory of gravitation in (2+1)-dimensional empty space-time is the weak-field limit of a non-linear subset curvature theory in (2+1)-dimensional space-time. To the best of our knowledge[7,10,11], the formulation of hidden nonlinearity in the weak field sound wave in (2+1)-dimensional space-time has not been done yet.

The writing structure of this article is as follows. In Sect. II, we discuss in brief the sound wave and its relation with Newton's second law. In Sect. III, we discuss the Newtonian limit. In Sect. IV, the weak-field limit of gravitation is discussed and a time-time component of the Ricci curvature tensor is derived. In Sect. V, we discuss subset fields property and maps $S^3 \rightarrow S^2$. In Sect. VI, in analogy to Hopf map, we define a subset curvature. In Sect. VII, we formulate non-linear and linearized Ricci theories using subset curvature. In Sect.VIII, we discuss potential and Clebsch variables. In Sect. IX, hidden nonlinearity in Newton's second law of gravitation is formulated. In Sect. X, we formulate hidden nonlinearity in weak field sound wave. Discussion and conclusion are given in Sect. XI.

II. SOUND WAVE AND NEWTON'S SECOND LAW

Sound is a form of energy produced by vibrations of medium molecules[12]. The vibrational energy from a sound source causes pressure disturbances from the equilibrium state of the medium. This perturbance causes the small changes to the density and the pressure. Sound wave (the propagation of a sound) is a longitudinal mechanical wave[11]. A wave could be defined roughly as the physical disturbance produced at one point of space, propagated through space, and felt later on at another point[10]. In the case of sound wave, the disturbance could be pressure variations in the gas[10,11]. So, the propagating disturbance in sound wave is the propagating pressure variations. Pressure can be related to the force and the acceleration where the acceleration can be replaced by the gravitational acceleration (the gravitational field).

Let us consider the sound wave resulting from the pressure variations in the gas[10,11]. Imagine a piston at one end of a long tube filled with a compressible medium (fluid). Assume that the fluid is a continuous medium and ignore for the time being the fact that it is made up of molecules that are in continual random motion.

Also, assume that we have a uniform fluid pressure and uniform density in its interior. If the piston oscillates back and forth, a continuous train of compressions and rarefactions will travel along the tube[11]. We call this longitudinal wave as a pulse, compressional zone, traveling at speed v along the tube[11].

Under the static condition or the equilibrium state of the medium, the particle velocity, $\vec{v}$, constant density, ρ, and the pressure, $p(\vec{x},t)$, is $\vec{v}=0$, $\rho=\rho_0$, $p=p_0$, where ρ_0 is the static air density and p_0 is the static air pressure. The vibrational energy from a source of sound causes pressure disturbances from the equilibrium state of the medium. This perturbance causes small changes to the density and the pressure, $\rho=\rho_0+\Delta\rho, p=p_0+\Delta p$, where $\Delta\rho \ll \rho$ and $\Delta p \ll p$.

Let us apply Newton's second law to the fluid element while the perturbation is entering the compressional zone. The resultant force has magnitude[11]

$$\vec{F}=(p_0+\Delta p)\vec{A}-p_0\vec{A}=\Delta p\ \vec{A} \tag{1}$$

where Δp is a different of pressure, $\vec{A}$ is the cross-sectional area of the tube.

In the case of a small difference of pressure, we can replace Δp with dp then eq.(1) can be written as

$$\vec{F}=\vec{A}\ dp \tag{2}$$

The relation between this resultant force and the equation of motion of the volume element can be written as[10]

$$\vec{A}\ dp=-(\rho_0\ \vec{A}\ dx)\frac{\partial^2\xi}{\partial t^2} \tag{3}$$

where $\xi(\vec{x},t)$ is the displacement. The physical meaning of eq.(3) is that the gas at the left of our volume element pushes to the right with a force pA and the gas at the right pushes to the left with a force p_0A. By rearranging terms, eq.(3) can be written as

$$\frac{\partial p}{\partial x}=-\rho_0\frac{\partial^2\xi}{\partial t^2} \tag{4}$$

Without derivation, we write the wave equation for describing the propagation of a sound in (2+1)-dimensional space-time as[10]

$$\frac{\partial^2 p}{\partial t^2}=c_s^2\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right) \tag{5}$$

where c_s is the speed of sound in the medium.

III. THE NEWTONIAN LIMIT

In the general theory of relativity, the motion of test bodies in (3+1)-dimensional curved space-time is governed by the geodesic equation which can be written as[1,2]

$$\frac{d^2x^\alpha}{d\tau^2}+\sum_{\mu,\nu}\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \tag{6}$$

where $x^\alpha(\tau)$ is the world line (representing the trajectory) of the particle in global inertial coordinates, $\alpha,\mu,\nu=0,1,2,3$, $\Gamma^\alpha{}_{\mu\nu}$ is the Christoffel symbols, and τ is the proper time. We consider inertial coordinates here to refer to frames of reference that are not accelerating. They are subject to a weak gravitational field only, allowing for a flat space-time approximation. Global means that these inertial coordinates cover the entire space-time.

In the Newtonian limit, we treat that the motion of a body is much slower than the speed of light. It has the consequence that the proper time may be approximated by the coordinate time, t. So, for the time-time components, $\mu,\nu=t$, we may approximate $dx^\mu/d\tau$, $dx^\nu/d\tau$, in the second term of eq.(6) as $(1,0,0,0)$. It means that the space-space components are vanish. Thus, eq.(6) becomes[2]

$$\frac{d^2x^\alpha}{dt^2}=-\Gamma^\alpha{}_{tt} \tag{7}$$

We have for the space components, $\alpha=1,2,3$[1,2]

$$\Gamma^\alpha{}_{tt}=\frac{\partial\phi}{\partial x^\alpha} \tag{8}$$

i.e. the Christoffel symbol, $\Gamma^\alpha{}_{tt}$, is related to the gradient of the gravitational (scalar) potential, ϕ. Here, again, due to the motion of a body being much slower than the speed of light, the time derivatives of ϕ have been neglected.

In the case of (1+1)-dimensional space-time, by substituting eq.(8) into (7), the motion of a body is governed by the equation[2]

$$\vec{a}=-\vec{\nabla}\phi \tag{9}$$

where $\vec{\nabla}$ is the gradient operator for 1-dimensional space and

$$\vec{a}=\frac{d^2\vec{x}}{dt^2} \tag{10}$$

is the acceleration of a body relative to global inertial coordinates of flat metric[2], the gravitational field (gravitational acceleration).

We see from eq.(9), in Newton's point of view, test bodies are in motion with acceleration or gravitational field. It means that there exists the gravitational forces act upon test bodies. The gravitational forces cause test bodies to orbit on "a curved line" in a flat space-time. On the other side, roughly speaking, we could say that eq.(6) shows the trajectories of test bodies following the geodesic "straight line" in the curved space-time. These points of view are the important difference between Einstein's general theory of relativity (6) and its Newtonian limit (9).

By using eq.(9), we can write Newton's second law of gravitation in (1+1)-dimensional space-time as

$$\vec{F}=m\ \vec{a}=-m\vec{\nabla}\phi \tag{11}$$

where $\vec{F}$ is the gravitational force, and m is mass. We see from eq.(11) that the difference in the gravitational potential shows the existence of acceleration or gravitational field. The existence of the gravitational force affects the test bodies to move with the acceleration. In analogy to the relation between the electromagnetic potential and the electromagnetic fields as shown e.g. in the Aharonov-Bohm effect[13], we might interpret that the gravitational potential is a more fundamental concept than the gravitational field.

IV. WEAK-FIELD LIMIT OF GRAVITATION

In the limit of weak gravitational fields, low velocities or static[6] (of gravitational sources), and small pressure, the general theory of relativity reduces to Newton's theory of gravitation[1]. In the case of the weak field, the metric tensor in (3+1)-dimensional space-time can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{12}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu}$ is small perturbation, $|h_{\mu\nu}| << 1$. Small perturbation have values[3]

$$h_{tt} = -2\phi,\ h_{t\mu} = h_{\mu t} = 0,\ h_{\mu\nu} = -2\ \delta_{\mu\nu}\ \phi \tag{13}$$

so the related metric can be written as

$$ds^2 = (1 - 2\phi)\ dx^2 - (1 + 2\phi)\ dt^2 \tag{14}$$

Linearization (by ignoring the non-linear terms of connection[6]) of the Ricci curvature tensor, due to the weak field, yields[1]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} \tag{15}$$

This equation is identical to Abelian field strength tensor in electrodynamics where the curvature (the Ricci tensor), $R_{\mu\nu}$, is identical to the field strength tensor, $F_{\mu\nu}$, and the connection (Christoffel symbol), $\Gamma^\alpha_{\mu\alpha}$, is identical to the gauge potential, A_μ.

The time-time component of Ricci curvature tensor (15) can be written as[1]

$$R_{tt} = \partial_\alpha \Gamma^\alpha_{tt} - \partial_t \Gamma^\alpha_{t\alpha} \tag{16}$$

where the second term in the right-hand side of (16) is assumed zero, $\partial_t {\Gamma^\alpha}_{t\alpha} = 0$ (due to the test body of the gravitational source moving very slowly or static). We consider that the choice of the time-time component of the Ricci curvature tensor is the simplest choice which relates to the Ricci curvature tensor in the case of the weak field and the Newtonian limit. Eq.(16) becomes

$$R_{tt} = \partial_\alpha \Gamma^\alpha_{tt} \tag{17}$$

where α, as we see in eq.(8), denotes the components of (3-dimensional) space.

By substituting (8) into (17) we obtain Newton's theory of gravitation written below[1,4,5]

$$R_{tt} = \nabla^2 \phi \tag{18}$$

where $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$ (div of grad) denotes the Laplace operator or Laplacian of (3-dimensional) space[2], and

$$\nabla^2 \phi = 4\pi\rho \tag{19}$$

is Poisson's equation[1,2], ρ is the mass density.

By substituting eq.(19) into eq.(18) we obtain Newton's theory of gravitation expressed as Newtonian field equation below[1]

$$R_{tt} = 4\pi\rho \tag{20}$$

Eq.(20) shows us that by choosing time-time components of the Ricci curvature tensor, we can recover Poisson's equation (19).

V. SUBSET FIELDS AND HOPF MAP $S^3 \to S^2$

Let us consider maps of subset fields (consisting of complex scalar fields) from a finite radius r to an infinite radius implying from the strong field to the weak field. A subset field has properties that, by definition, its value for a finite r depends on the magnitude and the direction of the position vector, $\vec{r}$, but for an infinite r it is well-defined[9] (it depends on the magnitude only). In other words, for an infinite r, a subset field is isotropic. The property of such subset fields can be interpreted as maps $S^3 \to S^2$[8] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres, respectively.

Let us discuss the maps above more formally. Assume that we have a subset field as a function of the position vector, $e^a(\vec{r})$, with a property that can be interpreted using the non-trivial Hopf map written below[8,9]

$$e^a(\vec{r}) : S^3 \to S^2 \tag{21}$$

This map $S^3 \to S^2$ can be classified in homotopy class labeled by the value of the corresponding Hopf index, integer number, and the topological invariant[8,9]. The other names of the topological invariant are the topological charge, and the winding number (the degree of a continuous mapping)[14]. The topological charge is independent of the metric tensor, it could be interpreted as energy[15]. The map (21) is a time-independent map where this time-independent problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[16].

We see from (21) that there exists (one) dimensional reduction in such maps. We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a subset field for an infinite r. The property of a subset field as a function of space seems likely in harmony with the property of space-time itself. Space-time could be locally anisotropic but globally isotropic (the distribution of matter energy in the universe is assumed to be homogeneous).

VI. SUBSET CURVATURE AND MAP OF THE GRAVITATIONAL THEORY $M^{3+1} \rightarrow M^{2+1}$

We propose that the curvature tensor, as it is identical to the field strength tensor, in an empty space-time could consist of the subset curvature, $e^a(\vec{r},t)$, where the superscript index a in $e^a(\vec{r},t)$ represents a set of indices that label the components of the subset curvature. These subset curvature, as components of the curvature tensor, have the property that can be interpreted using a map of the gravitational theory in (3+1) to (2+1)-dimensional space-time which is analog to the non-trivial Hopf map (21), can be written as below

$$e^a(\vec{r},t) : M^{3+1} \rightarrow M^{2+1} \tag{22}$$

where M denotes manifold. This map (22) is a time-dependent map. It differs from the Hopf map (21) which is time-independent. Throughout this article, we will work with the classical subset scalar curvature.

We see from (22), that there exists (one) dimensional space reduction in such a map. Analog to the non-trivial Hop map, we consider this dimensional reduction as a consequence of the isotropic (well-defined) property of subset curvature for an infinite r. The property of subset curvature as functions of space-time are the same as the property of space-time. Space-time could be locally anisotropic but globally isotropic.

VII. NON-LINEAR AND LINEARIZED RICCI THEORIES

By considering that the field strength tensor is identical to the curvature[13], we could write[8,9] the non-linear Ricci curvature tensor which its components satisfy the map (22) as follows

$$R^a_{\mu\nu} \approx \frac{\partial_\mu e^{a^*} \partial_\nu e^a - \partial_\nu e^{a^*} \partial_\mu e^a}{(1 + e^{a^*} e^a)^2} \tag{23}$$

where $e^a(\vec{r},t)$ is a subset of Ricci curvature tensor, and $e^{a^*}(\vec{r},t)$ is its complex conjugate. Eq.(23) is the non-linear curvature equation where the nonlinearity is shown by the $e^{a^*} e^a$ term in the denominator.

In the case of the weak field or empty space-time, the subset curvature are very small, $|e^{a^*} e^a| << 1$, so eq.(23) reduces to a linear curvature equation which can be written as below

$$R^a_{\mu\nu} \approx \partial_\mu e^{a^*} \partial_\nu e^a - \partial_\nu e^{a^*} \partial_\mu e^a \tag{24}$$

We see that this linearized Ricci curvature tensor (24) is identical to the electromagnetic field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

VIII. POTENTIAL AND CLEBSCH VARIABLES

Small perturbations of metric or linearized metric perturbations in eq.(12) take a role as "potentials" in the weak field or the linearized gravitation[3]. In the language of a wave, the small perturbations of metric can be written as[3]

$$h_{\mu\nu} = \rho_{\mu\nu}\ e^{i\vec{k}\cdot\vec{r}} \tag{25}$$

where $\rho_{\mu\nu}$ is amplitude, $\vec{k}$, $\vec{r}$ are wave and position vectors, respectively. In empty space-time, the space-time of a weak field, the amplitude is constant.

In analogy to (25), we consider the subset curvature (22) are related to perturbations of metric can be written as[17]

$$e^a(\vec{r},t) = \rho^a(\vec{r},t)\ e^{iq(\vec{r},t)} \tag{26}$$

where $\rho^a(\vec{r},t)$ is the amplitude, $q(\vec{r},t)$ is the phase. We could interpret the subset curvature as the perturbation or disturbance where the physical disturbance is the real part of the subset curvature[18]. The significant difference between $h_{\mu\nu}$ (25) and $e^a(\vec{r},t)$ (26) is $e^a(\vec{r},t)$ is valid for small and large perturbations of metric but $h_{\mu\nu}$ is valid for small perturbations of metric only. We could interpret that in the case of a weak field or empty space-time, $e^a(\vec{r},t)$ reduces to $h_{\mu\nu}$.

We could write the connection (the Christoffel symbol) as below

$$e^a_\nu = f^a\ \partial_\nu q \tag{27}$$

where the function of amplitude, f^a, could be written as

$$f^a = -1/\left\{2\pi[1 + (\rho^a)^2]\right\} \tag{28}$$

We see from eq.(27) that e^a_ν could be viewed as vector potential which is identical to the connection. Here, e^a_ν (27) is not a total derivative, otherwise it would be a pure gauge[16]. The subscript index ν in e^a_ν represents space-time coordinates.

We call the functions $f^a(\vec{r},t)$ and $q(\vec{r},t)$ the Clebsch variables[16] or Gaussian potentials[19,20]. These Clebsch variables are related to any divergenceless vector field[8]. An example of a divergenceless vector field is vorticity, $\vec{\omega}$, in hydrodynamics[20] or the magnetic field, $\vec{B}$, where $\vec{\nabla} \cdot \vec{B} = 0$. The Clebsch variables are not uniquely defined (many different choices are possible for them)[8].

By using eq.(27), the linearized Ricci curvature tensor (24) could be written as[16]

$$R^a_{\mu\nu} \approx \partial_\mu(f^a\ \partial_\nu q) - \partial_\nu(f^a\ \partial_\mu q) \tag{29}$$

This is the linearized Ricci curvature tensor written in terms of the Clebsch variables. Equation (29) is equivalent to eq.(15).

The time-time component of the linearized Ricci curvature tensor (29) could be written as

$$R^a_{tt} \approx \partial_\alpha(f^a\ \partial_\alpha q) - \partial_t(f^a\ \partial_t q) \tag{30}$$

where the index α denotes the space component (space coordinate). The second term on the right-hand side of (30) is equal to zero. It is because, in the Newtonian limit, it is considered that the speed of the body as the gravitational source is very slow compared to the speed of light. So eq.(30) becomes

$$R^a_{tt} \approx \partial_\alpha (f^a \ \partial_\alpha q) \tag{31}$$

Eq.(31) is the equation of Newton's theory of gravitation expressed using the Clebsch variables. The first ∂_α means divergence and the second is gradient. Roughly speaking, eq.(31) says that the source of the potential of gravitation is the curvature. Eq.(31) is equivalent to (17).

IX. HIDDEN NONLINEARITY IN NEWTON'S SECOND LAW OF GRAVITATION

By substituting eq.(8) into (7) we have

$$\frac{d^2 x^\alpha}{dt^2} = -\frac{\partial \phi}{\partial x^\alpha} = -\partial_\alpha \phi \tag{32}$$

where ∂_α denotes gradient. In the case of (1+1)-dimensional space-time, eq.(32) can be written as

$$\frac{d^2 \vec{x}}{dt^2} = -\frac{d\phi}{d\vec{x}} \tag{33}$$

This equation (33) is the same as eq.(9).

Let us define (10) using index notation, we have

$$\frac{d^2 x^\alpha}{dt^2} \equiv a_\alpha \tag{34}$$

where as we have mentioned previously, $\alpha = 1, 2$, denotes 2-dimensional space. By substituting eq.(34) into eq.(32), we obtain

$$a_\alpha = -\partial_\alpha \phi \tag{35}$$

By using index notation, eq.(18) can be written as

$$R_{tt} = \partial^\alpha \partial_\alpha \phi \tag{36}$$

where $\partial^\alpha = \partial / \partial x_\alpha$ denotes divergence. We see from eq.(36) that the divergence of the gradient of a scalar function is a scalar, so R_{tt} is a scalar.

By substituting eq.(35) into eq.(36) we obtain

$$R_{tt} = \partial^\alpha (-a_\alpha) = -\partial^\alpha a_\alpha \tag{37}$$

By integrating both sides of eq.(37) with respect to x_α, we find that

$$a_\alpha = -\int R_{tt} \ dx_\alpha \tag{38}$$

By substituting eq.(38) into (11) and replace $\vec{a}$ by a_α, $\vec{F}$ by F_α, we obtain

$$F_\alpha = -m \int R_{tt} \ dx_\alpha \tag{39}$$

where F_α is the gravitational (vector) force defined in (2+1)-dimensional space-time.

By substituting eq.(31) into (39) and replace R_{tt} by R^a_{tt} and F_α by F^a_α, we obtain

$$F^a_\alpha \approx -m \int \partial_\alpha (f^a \ \partial_\alpha q) \ dx_\alpha \tag{40}$$

Eq.(40) is the equation of Newton's second law of gravitation in (2+1)-dimensional space-time written using the Clebsch variables. Remember that mass, m, is a scalar, and the displacement, x_α, is a vector quantity.

X. HIDDEN NONLINEARITY IN WEAK FIELD SOUND WAVE

In index notation, the sound wave in (2+1)-dimensional space-time (5) can be written as

$$\partial^2_t p = c^2_s \ \partial^2_\alpha p \tag{41}$$

by keeping in mind that eq.(2) is equal to eq.(40), we have

$$dp \ A^a_\alpha = F^a_\alpha \approx -m \int \partial_\alpha (f^a \ \partial_\alpha q) \ dx_\alpha \tag{42}$$

where A^a_α denotes the area. So,

$$dp \approx -\frac{m}{A^a_\alpha} \int \partial_\alpha (f^a \ \partial_\alpha q) \ dx_\alpha \tag{43}$$

By substituting $m = \rho \ A^a_\alpha \ dx^\alpha$ into eq.(43) we obtain

$$\partial_\alpha p \approx -\rho \int \partial_\alpha (f^a \ \partial_\alpha q) \ dx_\alpha \tag{44}$$

where $\partial_\alpha p = \partial p / \partial x^\alpha$, ρ is a mass density.

By substituting (44) into (41), we have

$$\partial^2_t p = -c^2_s \ \partial_\alpha \left\{ \rho \int \partial_\alpha (f^a \ \partial_\alpha q) \ dx_\alpha \right\} \tag{45}$$

Eq.(45) is the linear equation of weak field sound wave in (2+1)-dimensional space-time which contains the hidden nonlinearity. What we mean by the field in sound wave (45) is pressure. The hidden nonlinearity is shown by the functions of f^a and q.

XI. DISCUSSION AND CONCLUSION

Roughly speaking, the general theory of relativity is Einstein's non-linear theory of gravitation, space, and time[21]. It describes the interplay between the local distribution of matter energy and the curvature of space-time[22]. In the limit of weak gravitational fields, low velocities of the test body or the gravitational sources, and small pressure, the general theory of relativity reduces to Newton's linear theory of gravitation[1]. Note that although the predictions of the general theory of relativity agree with those of Newton's theory of gravitation, the underlying point of view is radically different[2].

In the general theory of relativity point of view, the mass-energy of the sky object e.g. the Sun produces a curvature of the space-time. The Earth is in free motion without acceleration (no forces are acting upon the Earth). The Earth travels on a geodesic "straight line" of the curved space-time to orbit the Sun. It is shown by the geodesic equation (6). In Newton's point of view, the Sun creates the gravitational field (gravitational acceleration) as shown in eq.(9) that exerts a gravitational force upon the Earth. This gravitational force causes the Earth to orbit (on a curved line) the Sun rather than move in a straight line[2] in a flat space-time.

Newton's theory of gravitation, i.e. the curvature, R_{tt}, expressed using the Christoffel symbol (17), satisfies a linear field equation only, but subset curvature (expressed using the Clebsch variables) satisfy linear and non-linear field equations. Both satisfy a linear field equation in the case of the weak field of gravitation or empty space-time. It means that, in the case of the weak field or empty space-time, a non-linear subset curvature theory reduces to Newton's linear theory of gravitation i.e. the non-linear Ricci curvature tensor (23) reduces to the linearized Ricci curvature tensor (24). An empty space-time here means that there is no matter present and there is no physical fields exist except the weak gravitational field. This weak gravitational field does not disturb the emptiness. But other fields disturb the emptiness[23].

The linearized Ricci curvature tensor (24) is locally equivalent to eq.(15), but globally different. Eq.(15) is no longer valid globally. The difference between the subset curvature and the Ricci curvature tensor in empty space is global instead of local since the subset curvature obey the topological quantum condition but the Ricci curvature tensor does not.

In analogy to the Hopf map (21), we assume that subset curvature or components of the Ricci curvature tensor as a map of gravitational theory in (3+1) to (2+1)-dimensional space-time (22). This map (22) differs from a time-independent Hopf map (21). The time-independent Hopf map problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[16].

A map of gravitational theory in (3+1) to (2+1)-dimensional space-time implies that there exists (one) dimensional reduction in such a map (22). We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a subset field for an infinite r (infinite distance from the source) where the gravitational field is weak. It implies also that the linearized Ricci curvature tensor and the derived Newton's theory of gravitation can be formulated in (2+1)-dimensional space-time.

The related potential (27) which is identical to the Christoffel symbol, a connection, can be written using the Clebsch variables[8,16] or Gaussian potentials[19,20]. These Clebsch variables are related to any divergenceless vector field[8]. An example of a divergenceless vector field is vorticity, $\vec{\omega}$, in hydrodynamics[20] or the magnetic field, $\vec{B}$, where $\vec{\nabla} \cdot \vec{B} = 0$. The Clebsch variables are not uniquely defined (many different choices are possible for them)[8]. The related potential (22) could be viewed as vector potential. This vector potential is not a total derivative, otherwise, it would be a pure gauge[16].

By using the related potential (27), the Ricci curvature tensor (29) and its time-time component in the case of the weak-field and Newtonian limit (30), (31), can be formulated using the Clebsch variables. In turn, the time-time component of the Ricci curvature tensor (31) is useful when we construct Newton's second law of gravitation (40).

We could say that Newton's second law of gravitation (40) contains the hidden nonlinearity. The hidden nonlinearity is contained in the Clebsch variables. The hidden nonlinearity in Newton's second law of gravitation or, in general in Newton's theory of gravitation, has a consequence that sound wave which can be understood in terms of Newton's law[7], especially Newton's second law, has hidden nonlinearity also. It is shown by eq.(45) where eq.(45) is the linear equation of weak field sound wave in (2+1)-dimensional space-time which contains the hidden nonlinearity. What we mean by the field in sound wave (45) is pressure. The hidden nonlinearity is shown by the functions of f^a and q. Nonlinearity is the important character of the topology. The hidden nonlinearity of sound wave could have deep consequences as it could be related to the existence of the topological object (a gravitational knot) through the Chern-Simons action formulation[24].

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[1] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, W. H. Freeman and Company, 1973.

[2] Robert M. Wald, *General Relativity*, The University of Chicago Press, 1984.

[3] James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Pearson, 2014.

[4] Graham S. Hall, *The Geometry of Newton's and Einstein's Theories*, in Springer Handbook of Spacetime, Springer-Verlag, 2014.

[5] L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, 1989.

[6] M.P. Hobson, G.P. Efstathiou, A.N. Lasenby, *General Relativity: An Introduction for Physicists*, Cambridge University Press, 2006.

[7] Richard P. Feynman, Robert B. Leighton, Matthew Sands, *The Feynman Lectures on Physics. Volume I: Mainly Mechanics, Radiation, and Heat*, California Institute of Technology, 2010.

[8] Antonio F Ranada, *Topological electromagnetism*, J. Phys. A: Math. Gen. **25** (1992) 1621-1641. Antonio F. Ranada, *A Topological Theory of the Electromagnetic Field*, Letters in Mathematical Physics **18**: 97-106, 1989.

[9] Antonio F. Ranada, *A Topological Theory of the Electromagnetic Field*, Letters in Mathematical Physics **18**: 97-106, 1989.
[10] M. Alonso, E. J. Finn, *Fundamental University Physics*, Addison-Wesley, 1980.
[11] David Halliday, Robert Resnick, *Fundamentals of Physics*, John Wiley & Sons, Inc., 1976.
[12] Denny Hermawanto, *Application of Optical Sound Measurement in Microphone Calibration Method*, Doctoral thesis, Waseda University, 2021.
[13] Chen Ning Yang, *Topology and Gauge Theory in Physics*, International Journal of Modern Physics A, Vol. 27, No. 30 (2012) 1230035.
[14] Wikipedia, *Topological quantum number*.
[15] Miftachul Hadi, Hans Jacobus Wospakrik, *SU(2) Skyrme Model for Hadron*, https://arxiv.org/abs/1007.0888, 2010. Miftachul Hadi, Irwandi Nurdin, Denny Hermawanto, *Analytical Analysis and Numerical Solution of Two Flavours Skyrmion*, https://arxiv.org/abs/1006.5601, 2010.
[16] A.F. Ranada, A. Tiemblo, *A Topological Structure in the Set of Classical Free Radiation Electromagnetic Fields*, arXiv:1407.8145v1 [physics.class-ph] 29 Jul 2014.
[17] Miftachul Hadi, *Knot in weak-field geometrical optics*, https://osf.io/e7afj/, 2023, and all references therein.
[18] J.F. Nye, *Natural Focusing and Fine Structure of Light*, IOP Publishing Ltd, 1999.
[19] Roman Jackiw, *Diverse Topics in Theoretical and Mathematical Physics*, World Scientific, 1995.
[20] Roman Jackiw, *Lectures on Fluid Dynamics*, Springer, 2002.
[21] Sean Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Pearson, 2014.
[22] Wouter Merbis, *Chern-Simons–like Theories of Gravity*, Ph.D. Thesis, University of Groningen, 2014.
[23] P.A.M. Dirac, *General Theory of Relativity*, John Wiley & Sons, Inc., 1975.
[24] Miftachul Hadi, *Topological property of Newton's theory of gravitation*, https://osf.io/su3dp, 2023, and all references therein.