Simple arrangement in which it is possible to arrange the prime numbers in order to obtain a range of occupied consecutive positions greater than those defined by the Legendre Conjecture

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Abstract: The Legendre conjecture tells us that there always exists a prime number between N^2 and $(N+1)^2$. In this article, we will see that it is possible to arrange prime numbers less than or equal to N in order to obtain a consecutive number of occupied positions greater than the interval from N^2 to $(N+1)^2$. It is important to note that this result does not define a counterexample to the Legendre conjecture, but it represents an interesting theoretical result that can help us resolve many of the conjectures regarding the gap between two prime numbers.

The aim of this article is to find the arrangement in which prime numbers less than or equal to a number N occupy the greatest possible number of consecutive positions. Solving this problem is fundamental in understanding what the maximum gap is between two prime numbers and therefore being able to prove the conjectures relating to this problem: such as the Legendre conjecture.

To do this, we analyse two arrangements of numbers in which zero is the central value. In this way, the numbers to the right and left of the zero are symmetrical.

The first arrangement of length 4N is as follows:

-N –N+1...............-3 -2 -1 0 1 2 3.................... N-1 N

Note that in this arrangement there are a number of free positions, not occupied by prime numbers less than or equal to N, equal to the number of powers of 2 plus the two positions occupied by the values 1 and -1. Therefore, the number of free positions will be equal to $2NP2 + 2$ with NP2 the number of powers of the two minors equal to N.

The purpose of this arrangement is to understand the maximum number of consecutive positions that prime numbers less than N can occupy. **For this reason, the values in the arrangement are always associated with odd numbers.** Thus, the interval from -N to N has a length equal to 4N. **The zero of this arrangement is the product of the odd prime numbers <= N.**

First Arrangement Zero Position $= p_1 \cdot p_2 \dots \dots p_N \; with \; p_1 = 3$

Example: With N=7 the zero falls in the value $3 \cdot 5 \cdot 7 = 105$, the length interval 4N is as follows:

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
91	93	95	97	99	101	103	105	107	109	111	113	115	117	119

As we can see, the positions not occupied by prime numbers less than or equal to 7 are the following 6: -4 -2 -1 1 2 4.

The second arrangement is obtained by removing the even numbers from the previous arrangement.

-N –N+2..............–1 1 .... N-2 N

In this case, the length of the interval is halved, becoming 2N. In this arrangement there are always two numbers, -1 and 1, which are not occupied. So, unlike the previous case, where free positions increase as the value of N increases, in this case, free positions remain constant. **The zero of this arrangement is the product of the prime numbers <= N**.

Zero Position Second Arrangement $= p_1 \cdot p_2 \dots \dots p_N \; with \; p_1 = 2$

Example with N=7 the zero falls in the value $2 \cdot 3 \cdot 5 \cdot 7 = 210$, the length interval 2N is as follows:

-7	-5	-3	-1	1	3	5	7
203	205	207	209	211	213	215	217

As we can see, the positions not occupied by prime numbers less than or equal to 7 are 2: -1 1.

Legendre's conjecture states that there is always a prime number between n^2 and $(n+1)^2$. We note that the second arrangement, having two free positions and having a length of 2N, represents an arrangement in which the Legendre conjecture is true.

On the other hand, the first arrangement having a length of 4N has the possibility of generating a sequence of numbers in which more than 2N consecutive positions are occupied. In this case, we will have a situation in which Legendre's conjecture may no longer be valid.

This situation occurs when N > 103. In fact, with N = 103 the total free positions are 14 but of these 14 only 12 are included in the length range 2N which goes from -52 to 52. These 12 positions can be occupied by prime numbers greater than 52 and less than or equal to 103. The prime numbers with this feature are 12. Thus, by moving these 12 prime numbers from zero, we can fill all the free positions in the 2N length range of -52 to 52.

Figure 1: shows that in the case of N=103 it is possible to occupy 2N consecutive positions using prime numbers less than <= 103.

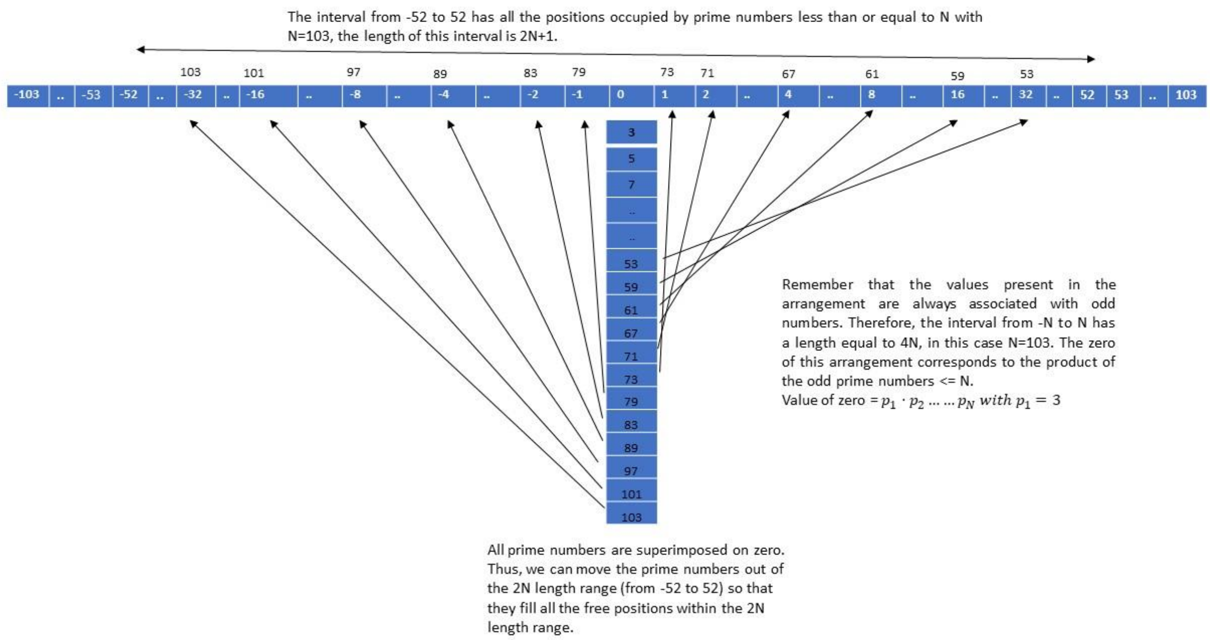

Figure 1: when N=103 we can occupy 2N consecutive positions using prime numbers <= 103

So, in summary, the second arrangement of length 2N turns out to be the optimal position for low values of N. When the values N exceeds 103, the arrangement in which the largest number of consecutive positions can be occupied is the first arrangement.

It is also interesting to study these two arrangements in the asymptotic case with N approaching infinity. In this case, the first arrangement is the one in which the prime numbers occupy the greatest number of non-overlapping positions. In fact, we know that a prime number p occupies a position not overlapping with any other odd prime number only if it is multiplied by itself or by a power of 2. In this case, the number of free positions is equal to the powers of 2 plus 1 Furthermore, since this arrangement is symmetrical these values must be multiplied by two. In the case of the second arrangement the prime numbers occupy non-overlapping positions only when they are multiplied by themselves. Therefore, compared to the previous case, the prime numbers appear to be more overlapping. However, a result is obtained that may be paradoxical. In fact, in this case, the positions not occupied by prime numbers are only two 1 and -1. So, in this arrangement there are fewer free positions than in the previous case despite the prime numbers being more overlapping with each other. Obviously, this paradoxical result is due to the divergence of the value of N to infinity. However, this asymptotic analysis is important because it tells us that with N tending to infinity the first arrangement is the one with which the prime numbers occupy the greatest number of non-overlapping positions and therefore it is the position in which the prime numbers less than and equal to a value of N (with large N) occupy the greatest number of positions. In fact, it is also what is observed experimentally when N is greater than 103.

It is experimentally seen that by increasing N the number of consecutive positions occupied by prime numbers less than or equal to N tends to 4N. So, from a theoretical point of view, it is possible to arrange prime numbers <= N in such a way as to violate Legendre's conjecture.

However, this result does not imply that Legendre's conjecture is wrong because this arrangement occurs with a period equal to $p_1 \cdot p_2 \dots \dots p_{Nm} con\ p_{Nm} < \frac{N}{2} e\ p_1 = 3$. So, at a much larger value than N^2. In fact, we know that in the interval from N^2 to $(N+1)^2$ the occupied positions can be found by considering only the prime numbers greater than or equal to N.

However, it is interesting from a theoretical point of view to know that it is possible to arrange prime numbers in such a way that they occupy a greater range than that defined by Legendre's conjecture.

In this article "The importance of finding the upper bounds for prime gaps in order to solve the twin primes conjecture and the Goldbach conjecture" I show how these two conjectures can be reformulated in such a way that their resolution depends on the maximum number of consecutive positions that prime numbers can occupy. Consequently, the search for the arrangement in which prime numbers less than or equal to a given value of N occupy the greatest number of consecutive positions is fundamental to the resolution not only of the Legendre conjecture but also of the twin prime conjecture and the Goldbach conjecture.

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