Unequal Volumetric and Shear Modulus in Einstein Field Equations

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Abstract

In the Einstein Field Equations space-time undergoes volumetric and shear deformations due to presence of matter as described by the stress-energy tensor. In the Einstein field equations the modulus associated to those two type of deformations is identical. In this paper we show that, in case this assumption is removed, proper solutions for the space-time metric can still be derived.

Key Words: Gravity, Elastostatics.

1 Introduction

The Einstein field equations describing curvature of space-time are the following:

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the curvature scalar and $T_{\mu\nu}$ is the stress–energy tensor.

Now suppose that $g_{\mu\nu}$ is the solution of the field equations for a given $T_{\mu\nu}$. We know that the trace of $g_{\mu\nu}$ is related to volumetric deformations in space-time. We can therefore represent $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{g}_{\mu\nu} \tag{2}$$

Where $\bar{g}_{\mu\nu}$ is the diagonal matrix with non vanishing elements equal to:

$$\bar{g}_{\mu\mu} = g_{\mu\mu} \tag{3}$$

and

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} \tag{4}$$

In the above decomposition of the metric, the tensor $\bar{g}_{\mu\nu}$ represents volumetric deformations and the tensor $\hat{g}_{\mu\nu}$ represents shear deformations of space-time. Note that $\bar{g}_{\mu\nu}$ has the same trace of $g_{\mu\nu}$ while $\hat{g}_{\mu\nu}$ is traceless.

In analogy with elastostatic, suppose now that deformation of space-time is proportional to pressure by a Young module Y. Multiplying both sides of equations (1) for our Young module, we have:

$$YR_{\mu\nu} + \frac{1}{2}R(Y\bar{g}_{\mu\nu} + Y\hat{g}_{\mu\nu}) = Y\frac{8\pi G}{c^4}T_{\mu\nu} \tag{5}$$

Roughly speaking, when matter and energy are present, they produce a pressure that, induces deformations in space-time. The terms in parenthesis represent the metric that conform to the tensor $T_{\mu\nu}$ with one term responsible for volumetric deformations and the other term for shear deformations.

However, this term alone is not enough because we would not get deferential equations but only an instantaneous deformation where $T_{\mu\nu} \neq 0$ and that does not take into account the propagations

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of the fields when $T_{\mu\nu}$ changes. We need an additional terms $R_{\mu\nu}$ in the equation and a $\frac{1}{2}R$ in front to the other terms to make the equations differential and compliant with continuity equations for energy-matter and momentum. This is because:

$$D_\mu \left(R^{\mu\nu} + \frac{1}{2} R(\bar{g}^{\mu\nu} + \hat{g}^{\mu\nu}) \right) = 0 \tag{6}$$

Where D_μ is the covariant derivative. Note that in the above equations the Young moduli for volumetric and shear deformations are identical.

2 Different Volumetric and Shear Moduli

Suppose now that the Young modulus for shear deformation E is different from the Young modulus for volumetric deformations. Equation (7) becomes:

$$Y\tilde{R}_{\mu\nu} + \frac{1}{2}\tilde{R}(Y\bar{g}_{\mu\nu} + E\hat{g}_{\mu\nu}) = Y\frac{8\pi G}{c^4}T_{\mu\nu} \tag{7}$$

Where now we need a new tensor $\tilde{R}_{\mu\nu}$ and a new scalar $\tilde{R}$, because equation (6) is not valid any more. Dividing by Y we have:

$$\tilde{R}_{\mu\nu} + \frac{1}{2}\tilde{R}(\bar{g}_{\mu\nu} + \epsilon\hat{g}_{\mu\nu}) = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{8}$$

where $\epsilon = E/Y$. Now we define a new tensor $\tilde{g}_{\mu\nu}$ as:

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\hat{g}_{\mu\nu} \tag{9}$$

and equation (8) becomes:

$$\tilde{R}_{\mu\nu} + \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{10}$$

It is now clear which tensor $\tilde{R}_{\mu\nu}$ and a new scalar $\tilde{R}$ we have to use in the above equation. They are simply the Ricci tensor and the scalar curvature relevant to the modified metric $\tilde{g}_{\mu\nu}$. By doing so, equation (6) is verified again.

We conclude that, in order to find the metric of space-time $g_{\mu\nu}$ when the shear modulus E is different from the volumetric modulus Y, we have to solve the very same equation (1) and find the metric $\tilde{g}_{\mu\nu}$ which is not the final metric we are looking for.

However, from the metric $\tilde{g}_{\mu\nu}$, by using equations (9), equation (3) and equation (4), we can easily find $g_{\mu\nu}$.

3 Observational Data

It is difficult to say if the hypotheses that $E \neq Y$ makes sense or not and if it agrees with some observational data. We are sure there are several cases where this hypotheses may be used and tested.

Among all, we cite an example. In recent years, many paper have been written to explain the anomalies observed in the measurements of galaxy rotation curves by means of frame dragging in the galaxy disks (see [1]). However, in these studies, the frame dragging is too week to explain the above curves. In this case, where the relative shear modules ϵ is greater then 1, in equation (10), this would induce an higher frame dragging that would help to explains the galaxy rotation curves without or partially avoiding the introduction of dark matter.

References

[1] F. Re, M. Galoppo. *On GR dragging and effective galactic dark matter.* https://arxiv.org/abs/2403.03227 (2024)