Kochański's approximation of Pi

Edgar Valdebenito

March 18, 2024

ABSTRACT

The problem of the exact rectification of a circle cannot by solved by classical geometry. Many approximate methods have been developed. Such an elegant one is Kochański's construction.

I. Introduction: Adam Kochański (1631-1700).

In his 1685 paper "Observationes cyclometricae" published in Acta Eruditorum, Adam Kochański presented an approximate ruler-and-compass construction for rectification of the circle.(see fig. 1):

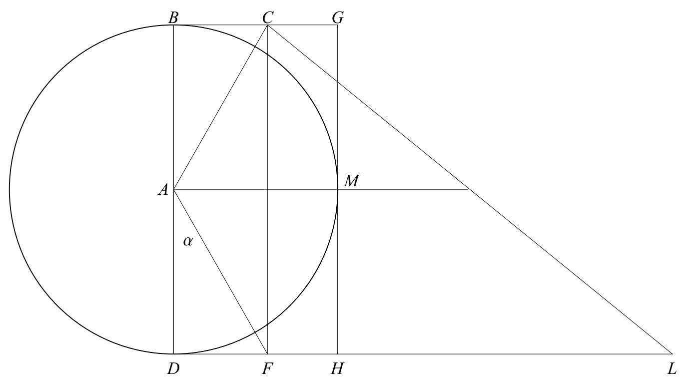

fig. 1. Kochański construction

- Consider for simplicity the unit circle (radius is 1): |AD| = |AB| = 1
- |CF| = |BD| = 2
- $\angle DAF = \angle BAC = \alpha = \pi/6$
- |DL| = 3
- By the construction: $|FL| = 3 - \tan\left(\frac{\pi}{6}\right) = 3 - \frac{1}{\sqrt{3}}$
- From Pythagoras' theorem: $|CL| = \sqrt{|CF|^2 + |FL|^2} = \sqrt{\frac{40}{3} - 2\sqrt{3}} = 3.1415333 .. \approx \pi$

II. Pi Formulas via Kochański's approximation

Entry 1. for $m = 1, 2, 3, 4, ...$, we have

$$\pi = \sqrt{\frac{40}{3} - 2\sqrt{3}} + \sqrt{\frac{40}{3} - 2\sqrt{3}} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{20 - 3\sqrt{3}}{3 \cdot 2^{2m+1}} \right)^n + 2^{m+1} \tan^{-1}\left(\frac{2^m r_m \sqrt{6} - \sqrt{20 - 3\sqrt{3}}}{2^m \sqrt{6} + \sqrt{20 - 3\sqrt{3}}\, r_m} \right)$$

where

$$r_m = \tan\left(\frac{\pi}{2^{m+1}}\right) \ , \ m = 1, 2, 3, 4, \ ...$$

$$r_1 = 1 \ , \ r_2 = \sqrt{2} - 1 \ , \ r_3 = \sqrt{\frac{2 - \sqrt{2 + \sqrt{2}}}{2 + \sqrt{2 + \sqrt{2}}}}$$

$$r_4 = \sqrt{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \quad , \quad r_5 = \sqrt{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$$

Entry 2. for $a = 20 - 3\sqrt{3}$, we have

$$\pi = \sqrt{\frac{40}{3} - 2\sqrt{3}} + 4\tan^{-1}\left(-1 + \frac{2}{1+}\,\frac{2\sqrt{6a}}{1\cdot 24-}\,\frac{a}{3-}\,\frac{a}{5\cdot 24-}\,\frac{a}{7-}\,\frac{a}{9\cdot 24-\ldots}\right)$$

Entry 3. for $a = 20 - 3\sqrt{3}$, we have

$$\pi = \sqrt{\frac{40}{3} - 2\sqrt{3}} + 2\tan^{-1}\left(\sqrt{\frac{6}{a}} - \frac{\sqrt{6a}}{6\cdot 3-}\,\frac{a}{5-}\,\frac{a}{6\cdot 7-}\,\frac{a}{9-}\,\frac{a}{6\cdot 11-}\,\frac{a}{13-\ldots}\right)$$

Entry 4.

$$\pi = \sqrt{\frac{40}{3} - 2\sqrt{3}} + 2\tan^{-1}\left(\sqrt{\frac{6}{20-3\sqrt{3}}} - \sqrt{\frac{6}{20-3\sqrt{3}}}\sum_{n=1}^{\infty}\frac{2^{2n}B_n}{(2n)!}\left(\frac{10}{3} - \frac{\sqrt{3}}{2}\right)^n\right)$$

where $B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \ldots\right\}$ are the Bernoulli numbers.

Entry 5.

$$\pi = K + 4\tan^{-1}\left(\frac{4-Ks}{4+Ks}\right)$$

where

$$K = \sqrt{\frac{40}{3} - 2\sqrt{3}}$$

$$s = \sum_{n=0}^{\infty}\frac{c_n}{(2n+1)!}\left(\frac{20-3\sqrt{3}}{24}\right)^n$$

$$c_n = (-1)^n - \sum_{k=1}^{n}(-1)^k\binom{2n+1}{2k}c_{n-k} \quad , \quad c_0 = 1\,, \quad n = 1, 2, 3, \ldots$$

$$c_n = \{1, 2, 16, 272, 7936, 353\,792, 22\,368\,256, \ldots\}$$

III. References

1. A. A. Kochański, Observationes cyclometricae ad facilitandam praxin accomodatae, Acta Eruditorum 4, 1685.
2. Z. Pawlikowska, Adam adamandy Kochański and his mathematical works, Rocznik Pol. Tow. Mat. 11, 1969. In Polish.
3. F. Lindemann, Über die Zahl π. Mathematische Annalen, 20, 1882.
4. E. W. Hobson, Squaring the Circle: A History of the Problem. Cambridge University Press, 2013.
5. B. Bold, Famous Problems of Geometry and How to Solve Them. New York: Dover, 1982.