A PROOF OF THE KAKEYA MAXIMAL FUNCTION CONJECTURE VIA BIG BUSH ARGUMENT

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Abstract. In this paper we reduce the Kakeya maximal function conjecture to the tube sets of unit measure. We show that the Kakeya maximal function is essentially monotonic. So by adding tubes we can reduce the conjecture to the case of unit measure tube set if we allow the technicality that there are possibly two tubes on the same direction. Then we proof the Kakeya maximal function conjecture from our lemma.

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1. Introduction

We define a line l_i as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and for all} \quad t \in \mathbf{R} \qquad y = a + xt\}$$

We define the δ-tubes as δ-neighborhoods of lines on $B(0,1)$ that intersect the ball $B(0,1/2)$

$$T_i^\delta := \{x \in B(0,1) || x - y| < \delta, \quad y \in l_i \quad \text{and} \quad B(0,1/2) \cap l_i \neq \emptyset\}.$$

So that lengths of our δ-tubes are bounded also from below. The order of intersection is defined as the number of tubes intersecting in an intersection. We define $A \lesssim B$ to mean that there exists a constant C_n depending only on n such that $A \leq C_n B$. We say that tubes are δ-separated if their angles are δ-separated. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,1)$ define a as it‘s center of mass. Define the Kakeya maximal function as
$f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T_\omega^\delta(a)} \int_{T_\omega^\delta(a)} |f(y)| \mathrm{d}y.$$

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In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f_\delta^*||_p \leq C_\epsilon \delta^{-n/p+1-\epsilon}||f||_p, \tag{1.1}$$

for all $\epsilon > 0$ and some $n \leq p \leq \infty$. A very important reformulation of the problem by Tao is the following. A bound of the form (1.1) follows from a bound of the form

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}(x)||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1.2}$$

for all $\epsilon > 0$, and for any set of $N \leq \delta^{1-n}$ δ-separated of δ-tubes. See for example [2] or [1]. It's enough to consider the case $p = n$ and the rest of the cases will follow via interpolation [1, 2]. In this paper any constant can depend on dimension n. Our main lemma is the following:

Lemma 1.1. *Let there be a $N \sim \delta^{1-n}$ δ-tubes that are δ-separated. Then we have*

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}(x)||_{n/(n-1)} \lesssim (\ln N)^{(n-1)/n}||\sum_{\omega'\in\Omega'} 1_{T_\omega(a_\omega)}(x)||_{n/(n-1)},$$

where Ω' is almost δ-separated with two tubes of the same direction and

$$\bigcup_{\omega'\in\Omega'} T_{\omega'}^\delta = \bigcup_{\omega\in\Omega} T_\omega^\delta(a_\omega) \cup \bigcup_{i=1}^N T_i^\delta(0). \tag{1.3}$$

Our main theorem is the following.

Theorem 1.2. *Let there be a $N \sim \delta^{1-n}$ δ-tubes that are δ-separated. Then we have that*

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}(x)||_{n/(n-1)} \lesssim (\ln N)^{2n/(n-1)}.$$

The case $n = 2$ of the Kakeya maximal function conjecture is well know to be true [1]. The case $n = 1$ is trivial.

2. Previously Known Results

We will use the following bound for the pairwise intersections of δ-tubes:

Lemma 2.1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n \cap B(0,1)$, we have*

$$|T_{\omega_i}^\delta(a) \cap T_{\omega_j}^\delta(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [1]. For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of Ω.

Lemma 2.2. *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [1].

3. The proof of the lemma

We assume that $N \sim \delta^{1-n}$ We also drop the δ-upper index and the center points a_i so we have

$$1 \sim \delta^{n-1}N \sim \sum_{i=1}^{N} |T_i| = \int \sum_{i=1}^{N} 1_{T_i}(x).$$

We define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \le \sum_{i=1}^{N} 1_{T_i}(x) \le 2^{k+1}\}.$$

So we have

$$\int_{E_{2^k}} \sum_{i=1}^{N} 1_{T_i}(x) \sim \sum_{i=1}^{N} \int_{E_{2^k}} 1_{T_i}(x) \sim \sum_{k} 2^k |E_{2^k}|. \tag{3.1}$$

However, we can also calculate

$$\sum_k \int_{E_{2^k}} \sum_{i=1}^{N} 1_{T_i}(x) \sim \sum_k \sum_{i=1}^{N} \int_{E_{2^k}} 1_{T_i}(x) \sim \sum_k \sum_{i=1}^{N} |E_{2^k} \cap T_i| \sim \sum_{i=1}^{N} |T_i| \sim \delta^{n-1}N. \tag{3.2}$$

We also notice that the number of k is less than $\sim \ln N$. Now, we have from (3.1) and from (3.2) that

$$\delta^{n-1}N \sim \sum_k 2^k |E_{2^k}|. \tag{3.3}$$

Next we use our big bush argument. We consider N δ-tubes that are δ-separated. Moreover, all the center points of the tubes are in the origin. This set $\bigcup_{j=1}^{N} T_j^{\delta}(0)$ is the so called big bush. It's clear that

$$|\bigcup_{j=1}^{N} T_j^{\delta}(0)| \sim \delta^{n-1}N,$$

because if $N \sim \delta^{1-n}$, the big bush covers the unit ball. However the number of tubes N only doubles if take the union with the original tube set! So we take the union

$$E' := \bigcup_{i=1}^{N} T_i^{\delta}(a_i) \cup \bigcup_{j=1}^{N} T_j^{\delta}(0),$$

and do another dyadic decomposition. We have then

$$\delta^{n-1}N \sim \sum_m 2^m |E'_{2^m}|. \tag{3.4}$$

Now if $x \in E_{2^k}$ then $x \in \bigcup_{m \ge k} E'_{2^m}$! This is the monotonicity condition. It follows because if some point x belongs to $\sim 2^k$ tubes then after adding more tubes x belongs to at least $\sim 2^k$tubes. So we have the key inequality

$$2^k |E_{2^k}|^{(n-1)/n} \lesssim (\sum_{m \ge k} 2^{mn/(n-1)} |E'_{2^m}|)^{(n-1)/n}. \tag{3.5}$$

It's clear via dyadic decomposition that

$$|| \sum_{\omega \in \Omega} 1_{T_\omega(a_\omega)}(x) ||_{n/(n-1)} \sim (\sum_k 2^{kn/(n-1)} |E_{2^k}|)^{(n-1)/n}.$$

So we have from (3.5) that

(3.6)

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}(x)||_{n/(n-1)} \sim (\sum_m 2^{kn/(n-1)}|E_{2^k}|)^{(n-1)/n} \lesssim (\ln N)^{(n-1)/n} \max_k 2^k |E_{2^k}|^{n/(n-1)}$$

$$\lesssim (\ln N)^{(n-1)/n} \sum_{m\geq k} (2^{mn/(n-1)}|E'_{2^m}|)^{(n-1)/n}.$$

So we are done proving our main lemma 1.1, because we can combine the above (3.6) with

$$(\sum_{m\geq k} 2^{mn/(n-1)}|E'_{2^m}|)^{(n-1)/n} \lesssim (\sum_m 2^{mn/(n-1)}|E'_{2^m}|)^{(n-1)/n} \sim ||\sum_{\omega'\in\Omega} 1_{T'_\omega(a_{\omega'})}(x)||_{n/(n-1)}.$$

4. The proof of the theorem

Next we use the lemma 1.1 to proof the theorem 1.2. We will assume the big bush condition (1.3) and we will prove

$$||\sum_{i=1}^{2N} 1_{T_i(a_i)}(x)||_{n/(n-1)} \lesssim (\ln N)^{(n-1)/n}. \tag{4.1}$$

We define for the δ-separated big bush $\bigcup_{i=1}^N T_i^\delta(0)$ that

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^N 1_{T_i(0)}(x) \leq 2^{k+1}\}.$$

It's a fact that

$$|B(0, 2^{-k/(n-1)})| \sim |E_{2^k}|. \tag{4.2}$$

We don't go in to details with above (4.2), but it follows from a sharp case of the lemma of Córdoba 2.1. In the case of (4.2) each of the N tubes intersects of order $\sim 2^k$ in length of $\sim 2^{-k/(n-1)}$. Integrating

$$2^m \leq \sum_{i=1}^{2N} 1_{B(0,1)}(x) 1_{T_i(a_i)}(x) \leq 2^{m+1}$$

over E'_{2^m} we have

$$2^m |E'_{2^m}| \sim \sum_{i=1}^{2N} |E'_{2^m} \cap T_i^\delta(a_i)|.$$

We now have a key inequality:

$$\begin{aligned} & 2^{-jn/(n-1)} \sim |B(0, 2^{-j/(n-1)})| \\ & \sim 2^{-j} \sum_{i=1}^N |T_i^\delta(0) \cap B(0, 2^{-j/(n-1)})| \\ & \leq 2^{-j} \sum_{i=1}^{2N} |T_i^\delta(a_i) \cap B(0, 2^{-j/(n-1)})| \\ & \lesssim N\delta^{n-1} 2^{-j} 2^{-j/(n-1)} \\ & \sim 2^{-jn/(n-1)}, \end{aligned} \tag{4.3}$$

where we used fact (4.2), that

$$2^j |E_{2^j}| \sim \sum_{i=1}^{N} |E_{2^j} \cap T_i^\delta(0)|$$

and only assumed the big bush condition (1.3). So it follows from (4.3) that for all dyadic $2^{-j/(n-1)}$ radius

$$\sum_{i=1}^{2N} |B(0, 2^{-j/(n-1)}) \cap T_i^\delta(a_i)| \sim \sum_{i=1}^{N} |B(0, 2^{-j/(n-1)}) \cap T_i^\delta(0)|. \tag{4.4}$$

Because by the big bush condition (1.3) we have that

$$1 \sim \sum_{i=1}^{2N} |T_i^\delta(a_i)| \sim \sum_{i=1}^{N} |T_i^\delta(0)|,$$

then for all dyadic $2^{-j/(n-1)}$ it follows from (4.4) that

$$\sum_{i=1}^{2N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(a_i)| \sim \sum_{i=1}^{N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(0)|.$$

So it follows that

$$\sum_{i=1}^{N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(0) \cap \{E'_{2^m}\}^c| + \sum_{i=1}^{N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(0) \cap E'_{2^m}|$$

$$\sim \sum_{i=1}^{2N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(a_i) \cap \{E'_{2^m}\}^c| + \sum_{i=1}^{2N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(a_i) \cap E'_{2^m}|.$$

Because from the big bush condition (3.5) it follows that

$$\sum_{i=1}^{N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(0) \cap E'_{2^m}| \leq \sum_{i=1}^{2N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(a_i) \cap E'_{2^m}|$$

and

$$\sum_{i=1}^{N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(0) \cap \{E'_{2^m}\}^c| \leq \sum_{i=1}^{2N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(a_i) \cap \{E'_{2^m}\}^c|,$$

it follows that

$$\sum_{i=1}^{N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(0) \cap E'_{2^m}| \sim \sum_{i=1}^{2N} |(B(0, 2^{-j/(n-1)}))^c \cap T_i^\delta(a_i) \cap E'_{2^m}|. \tag{4.5}$$

Because the above (4.5) holds for all balls of dyadic radius we have from above (4.5) that

$$\sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m}| \sim \sum_{i=1}^{2N} |T_i^\delta(a_i) \cap E'_{2^m}|. \tag{4.6}$$

From the above (4.6) the claim (4.1) is implied by the Kakeya maximal function conjecture for the δ-separated big bush $\bigcup_{i=1}^{N} T_i^\delta(0)$ [3]. From (4.6) the claim (4.1) is implied by

$$\sum_{i=1}^{N} |A \cap T_i^\delta(0)| \lesssim (\ln N)^{(n-1)/n} |A|^{1/n},$$

for all measurable sets A. Let

$$f := 1_A$$

be the indicator function of $A \subset B(0,1)$. Then the claim is implied by

$$\begin{aligned}\sum_{i=1}^{N} |A \cap T_i^\delta(0)| &= \int f(x) \sum_{i=1}^{N} 1_{T_i(0)}(x) \\ &\leq || \sum_{i=1}^{N} 1_{T_i^\delta(0)}(x))||_{n/(n-1)} ||f||_n \lesssim (\ln N)^{(n-1)/n} ||f||_n,\end{aligned}$$

which is implied by

$$|| \sum_{i=1}^{N} 1_{T_i^\delta(0)}(x)||_{n/(n-1)} \lesssim (\ln N)^{n/(n-1)}. \tag{4.7}$$

So we are essentially done, because we have reduced the claim to a standard example. We can prove the claim (4.1) different way also from (4.5). Now, some ball $B(0, 2^{-l/(n-1)})$ with dyadic radius has essentially the same measure that E'_{2^m} has. In other words

$$|E_{2^l}| \sim |B(0, 2^{-l/(n-1)})| \sim |E'_{2^m}|, \tag{4.8}$$

where, as defined before, $|E_{2^l}|$ is a level set of the big bush. We will now proof that

$$2^m |E'_{2^m}| \sim \sum_{i=1}^{2N} |T_i^\delta(a_i) \cap E'_{2^m}| \sim \sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m}| \lesssim 2^l |E'_{2^m}|. \tag{4.9}$$

Now let us note a key geometrical fact that

$$\sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m} \cap (B(0, 2^{-l/(n-1)}))^c| \lesssim 2^l |E'_{2^m}|, \tag{4.10}$$

because outside of $B(0, C_n 2^{-l/(n-1)})$ there aren't any origin centered tubes $T_i^\delta(0)$ intersecting on order greater than 2^l. This can be seen from the fact (4.2). So we have

$$\begin{aligned}\sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m}| &= \sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m} \cap B(0, 2^{-l/(n-1)})| + \sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m} \cap (B(0, 2^{-l/(n-1)}))^c| \\ &\lesssim \sum_{i=1}^{N} |T_i^\delta(0) \cap B(0, 2^{-l/(n-1)})| + \sum_{i=1}^{N} |T_i^\delta(0) \cap E'_{2^m} \cap (B(0, 2^{-l/(n-1)}))^c| \\ &\lesssim 2^{-l/(n-1)} N \delta^{n-1} + 2^l |E'_{2^m}| \\ &\sim 2^l |E_{2^l}|,\end{aligned}$$

where we used (4.10) and (4.8).

So (4.9) holds. So we have

$$2^m |E'_{2^m}| \lesssim 2^l |E'_{2^m}|,$$

So
$$2^m \lesssim 2^l.$$
Now the claim is straightforward. We are done for example from the dyadic decomposition of (4.1):
$$\ln N \sim \sum 2^{ln/(n-1)}|E_{2^l}| \gtrsim \sum 2^{mn/(n-1)}|E'_{2^m}| \sim ||\sum_{i=1}^{2N} 1_{T_i(a_i)}(x)||_{n/(n-1)}^{n/(n-1)},$$
where we used (4.8) and (4.9). So (4.1) holds and we are done proving the theorem 1.2.

References

[1] E.Kroc, *The Kakeya problem*, available at

http://ekroc.weebly.com/uploads/2/1/6/3/21633182/mscessay-final.pdf

[2] T. Tao, *Lecture Notes*, available at math.ucla.edu/ tao/254b.1.99s/ (1999)

[3] T. Tao *Blog Post*, available at https://terrytao.wordpress.com/2009/05/15/the-two-ends-reduction-for-the-kakeya-maximal-conjecture/ (2009)

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