*Article*

# Geometrized Vacuum Physics. Part II. Algebra of Signatures

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**Abstract:** This article is the second part of a scientific project under the general name "Geometrized vacuum physics". On the basis of the Algebra of Stignatures presented in the previous article [1], this article develops the main provisions of the Algebra of Signatures. Both of the above algebras are aimed at studying the properties of an ideal vacuum, but at the same time they are universal and can be applied in various branches of knowledge. It is shown that the signature of a quadratic form is related to the topology of the metric space for which the given quadratic form is a metric. Conditions are given under which an additive imposition of metric spaces with different topologies (or signatures) leads to a total Ricci flat space similar to a Calabi-Yau manifold. A spin-tensor representation of metrics with different signatures is considered and a Dirac bundle of quadratic forms is presented. This article does not contain physical applications of the Algebra of Signatures, but the potential power of this mathematical apparatus will be demonstrated in subsequent articles of this project.

**Resumen:** Este artículo es la segunda parte de un proyecto científico bajo el nombre general de "Física del vacío geometrizada". Sobre la base del Álgebra de Signaturas presentada en el artículo anterior [1], este artículo desarrolla las principales disposiciones del Álgebra de Signaturas. Las dos álgebras anteriores están dirigidas a estudiar las propiedades de un vacío ideal, pero al mismo tiempo son universales y se pueden aplicar en varias ramas del conocimiento. Se muestra que la firma de una forma cuadrática está relacionada con la topología del espacio métrico para el cual la forma cuadrática dada es una métrica. Se dan condiciones bajo las cuales una imposición aditiva de espacios métricos con diferentes topologías (o firmas) conduce a un espacio plano total de Ricci similar a una variedad de Calabi-Yau. Se considera una representación de tensor de espín de métricas con diferentes firmas y se presenta un conjunto de formas cuadráticas de Dirac. Este artículo no contiene aplicaciones físicas del álgebra de firmas, pero el poder potencial de este aparato matemático se demostrará en artículos posteriores de este proyecto.

**Keywords:** vacuum; geometrized vacuum physics; signature; algebra of signature

**Palabras clave:** vacío; física del vacío geometrizado; signatura; álgebra de signatura

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## 1 Introduction

This article is the second of a series of articles under the general title "Geometrized vacuum physics ", and is devoted to the presentation of the foundations of the Algebra of signatures.

In the first article [1], a local volume of ideal vacuum was considered, in which, by means of probing with mutually perpendicular light rays with a wavelength $\lambda_{m,n}$ (from the subrange $\Delta\lambda = 10^m \div 10^n$ cm), a $3D_{m,n}$-cubic lattice was obtained (see Figure 1, or Figure 5 in [1])

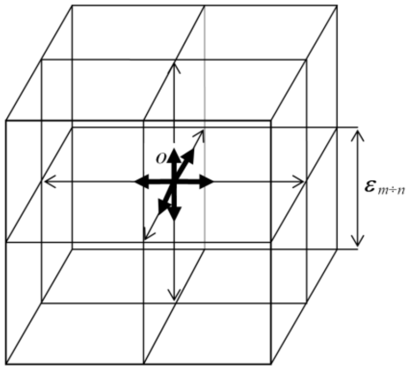



**Figure 1.** Non-curved 3D light lattice of $\lambda_{m,n}$-vacuum, revealed from the "vacuum" (emptiness) by means of mutually perpendicular monochromatic rays of light with a wavelength $\lambda_{m,n}$. The cells of such a lattice are cubes with edge length $\varepsilon_{m\div n} \sim 10^2 \cdot \lambda_{m,n}$.

The three-dimensional extent revealed from the void using such a luminous $3D_{m,n}$ cubic lattice is called in [1] $\lambda_{m,n}$-vacuum or $3D_{m,n}$-landscape.

In § 3 of the article [1], it was found that the number of orthogonal 3-bases that originate at the central point $O$ (see Figure 1), taking into account the direction of the time axis, is 16

3-bases shown in Figure 2 correspond to sixteen types of affine spaces that can be characterized by the corresponding signatures (see §4 and Table 1 in [1]). These sixteen stignatures of affine spaces form the stignature matrix (3) in [1]:

$$stign(e_i^{(a)}) = \begin{pmatrix} \{++++\} & \{+++-\} & \{-++-\} & \{++-+\} \\ \{---+\} & \{-+++\} & \{--++\} & \{-+-+\} \\ \{+--+\} & \{++--\} & \{+---\} & \{+-++\} \\ \{--+-\} & \{+-+-\} & \{-+--\} & \{----\} \end{pmatrix} \qquad (2)$$

Some properties of this matrix and the foundation of the Algebra of stignatures are described in [1].

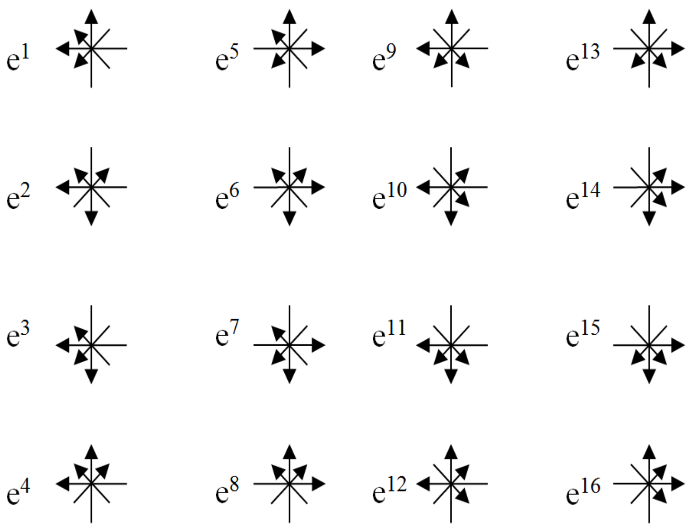



**Figure 2.** Sixteen 4-bases starting at point $O$ [1].

In this article, a transition is made from sixteen affine spaces with stignatures (2), which originate at the point $O$, to 256 × 4 = 1024 metric spaces, which intersect at the same point, under the condition of the "vacuum balance".

*The conditions of "vacuum (i.e. zero) balance" were formulated in the article [1,2]: "If something is born from a vacuum, it is necessarily in a mutually opposite form (particle – antiparticle, convexity – concavity, wave – anti-wave, etc.), and on average remains equal to zero".*

Moreover, each metric space is characterized by the corresponding signature. The totality of these signatures forms a matrix of signatures, the property of which is investigated in this article.

Also, in this second part of the "Geometrized Vacuum Physics" the foundations of the Algebra of Signatures are laid, which can be applied in various branches of scientific knowledge.

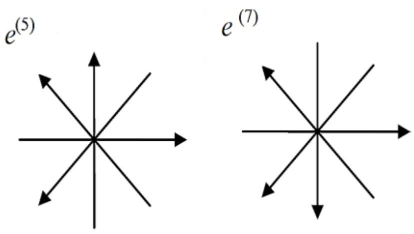
Together, the Algebra of Stignatures and the Algebra of Signatures form a single universal mathematical apparatus that can serve as the basis for describing and explaining many physical phenomena that were previously difficult to comprehend. The application of this apparatus to solving various physical problems will be presented in the following articles of the proposed project.

## 2 Materials and Method

### 2.1 Transition from 16 affine spaces to 256 metric spaces

We pass from the sixteen affine spaces with 4-bases shown in Figure 2 and their corresponding signatures (2) to metric spaces.

To do this, as an example, out of sixteen 4-bases (see Figure 2), we choose the 4-basis $\mathbf{e}_i^{(7)}(\mathbf{e}_0^{(7)},\mathbf{e}_1^{(7)},\mathbf{e}_2^{(7)},\mathbf{e}_3^{(7)})$ with signature {+ + + –} and 4-basis $\mathbf{e}_i^{(5)}$ $(\mathbf{e}_0^{(5)}, \mathbf{e}_1^{(5)}, \mathbf{e}_2^{(5)}, \mathbf{e}_3^{(5)})$ with signature {+ + + +} (see Figure 3)




a) {+ + + +}  b) {+ + + –}

**Figure 3.** Two 4-bases with different stignatures.

Let's define two 4-vectors in affine spaces with 4-bases $\mathbf{e}_i^{(5)}$ and $\mathbf{e}_i^{(7)}$

$$d\mathbf{s}^{(7)} = \mathbf{e}_i^{(7)}dx_i^{(7)} = \mathbf{e}_0^{(7)}dx_0^{(7)} + \mathbf{e}_1^{(7)}dx_1^{(7)} + \mathbf{e}_2^{(7)}dx_2^{(7)} + \mathbf{e}_3^{(7)}dx_3^{(7)}, \tag{3}$$

$$d\mathbf{s}^{(5)} = \mathbf{e}_i^{(5)}dx_i^{(5)} = \mathbf{e}_0^{(5)}dx_0^{(5)} + \mathbf{e}_1^{(5)}dx_1^{(5)} + \mathbf{e}_2^{(5)}dx_2^{(5)} + \mathbf{e}_3^{(5)}dx_3^{(5)}, \tag{4}$$

where $dx_i^{(k)}$ is the $i$-th projection of the 4-vector $d\mathbf{s}^{(k)}$ onto the $x_i^{(k)}$ axis, whose direction is determined by the basis vector $\mathbf{e}_i^{(k)}$.

Let's find the scalar product of 4-vectors (48) and (49)

$$\begin{aligned} ds^{(5,7)\,2} &= d\mathbf{s}^{(5)}d\mathbf{s}^{(7)} = \mathbf{e}_i^{(5)}\mathbf{e}_j^{(7)}dx^i\,dx^j = \\ &= \mathbf{e}_0^{(5)}\mathbf{e}_0^{(7)}dx_0dx_0 + \mathbf{e}_1^{(5)}\mathbf{e}_0^{(7)}dx_1dx_0 + \mathbf{e}_2^{(5)}\mathbf{e}_0^{(7)}dx_2dx_0 + \mathbf{e}_3^{(5)}\mathbf{e}_0^{(7)}dx_3dx_0 + \\ &+ \mathbf{e}_0^{(5)}\mathbf{e}_1^{(7)}dx_0dx_1 + \mathbf{e}_1^{(5)}\mathbf{e}_1^{(7)}dx_1dx_1 + \mathbf{e}_2^{(5)}\mathbf{e}_1^{(7)}dx_2dx_1 + \mathbf{e}_3^{(5)}\mathbf{e}_1^{(7)}dx_3dx_1 + \\ &+ \mathbf{e}_0^{(5)}\mathbf{e}_2^{(7)}dx_0dx_2 + \mathbf{e}_1^{(5)}\mathbf{e}_2^{(7)}dx_1dx_2 + \mathbf{e}_2^{(5)}\mathbf{e}_2^{(7)}dx_2dx_2 + \mathbf{e}_3^{(5)}\mathbf{e}_2^{(7)}dx_3dx_2 + \\ &+ \mathbf{e}_0^{(5)}\mathbf{e}_3^{(7)}dx_0dx_3 + \mathbf{e}_1^{(5)}\mathbf{e}_3^{(7)}dx_1dx_3 + \mathbf{e}_2^{(5)}\mathbf{e}_3^{(7)}dx_2dx_3 + \mathbf{e}_3^{(5)}\mathbf{e}_3^{(7)}dx_3dx_3. \end{aligned} \tag{5}$$

For the case under consideration, the scalar products of basis vectors $\mathbf{e}_i^{(5)}\mathbf{e}_j^{(7)}$ are:

$$\begin{aligned} &\text{for} \quad i = j \quad \mathbf{e}_0^{(5)}\mathbf{e}_0^{(7)} = 1, \quad \mathbf{e}_1^{(5)}\mathbf{e}_1^{(7)} = 1, \quad \mathbf{e}_2^{(5)}\mathbf{e}_2^{(7)} = 1, \quad \mathbf{e}_3^{(5)}\mathbf{e}_3^{(7)} = -1, \\ &\text{for} \quad i \neq j \quad \text{all} \quad \mathbf{e}_i^{(5)}\mathbf{e}_j^{(7)} = 0. \end{aligned} \tag{6}$$

In this case, Ex. (5) becomes the quadratic form

$$ds^{(5,7)2} = dx_0dx_0 + dx_1dx_1 + dx_2dx_2 - dx_3dx_3 = dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2, \quad \text{with signature } (+++-). \tag{7}$$

Recall that the "signature" (the term of general relativity) is an ordered set of signs in front of the corresponding terms of the quadratic form.

To determine the signature of a metric space with metric (7), instead of performing the scalar product of vectors (5), it suffices to multiply the signs of the signatures of the 4-bases shown in Fig.3:

$$\begin{array}{c} \{++++\} \\ \underline{\{+++-\}} \\ (+++-)_\times \end{array} \tag{8}$$

In the numerator of the rank (8), the multiplication of signs in each column is performed according to the rules

$$\{+\} \times \{+\} = \{+\}; \qquad \{-\} \times \{+\} = \{-\}, \tag{9}$$

the result of such multiplication is written in the denominator (under the line) of the same column. The performance of actions according to these rules will be called rank multiplication.

Just as it was done with the vectors $\boldsymbol{ds}^{(5)}$ and $\boldsymbol{ds}^{(7)}$ {see Exs. (3) – (9)}, we scalarly multiply vectors from all 16 affine spaces with 4-bases, shown in Figure 3. As a result, we obtain 16 × 16 = 256 metric 4-spaces with 4-metrics of the form

$$ds^{(a,b)2} = \mathbf{e}_i^{(a)}\mathbf{e}_j^{(b)}dx^{i(a)}dx^{j(b)}, \tag{10}$$

where $a = 1, 2, 3, \dots, 16$; $b = 1, 2, 3, \dots, 16$.

The signatures of these 16 × 16 = 256 metric 4-spaces can be determined, similarly to (8), by rank multiplications of the signs of the signatures of the corresponding affine spaces, for example:

$$\begin{array}{cccc}
\{+-++\} & \{++++\} & \{-+++\} & \{++++\} \\
\underline{\{+++-\}} & \underline{\{+-+-\}} & \underline{\{+++-\}} & \underline{\{-++-\}} \\
(+-+-)_\times & (+-+-)_\times & (-++-)_\times & (-++-)_\times \\
 & & & \\
\{+--+\} & \{++-+\} & \{-+++\} & \{+-+-\} \\
\underline{\{+++-\}} & \underline{\{-++-\}} & \underline{\{-++-\}} & \underline{\{+-+-\}} \\
(+---)_\times & (-+--)_\times & (+++-)_\times & (++++)_\times \\
 & & & \\
\{+---\} & \{++-+\} & \{-+-+\} & \{+-++\} \\
\underline{\{+++-\}} & \underline{\{-+--\}} & \underline{\{--+-\}} & \underline{\{+-+-\}} \\
(+--+)_\times & (-++-)_\times & (+---)_\times & (+++-)_\times \\
\dots & \dots & \dots & \dots \\
\{+++-\} & \{-+--\} & \{-++-\} & \{+--+\} \\
\underline{\{--+-\}} & \underline{\{+-+-\}} & \underline{\{+-+-\}} & \underline{\{-++-\}} \\
(--++)_\times & (---+)_\times & (--+-)_\times & (----)_\times
\end{array} \tag{11}$$

The point $O$ (see Figure 1) is the intersection point of all 256 metric 4-spaces with 4-metrics (10) and the corresponding signature (11).

A set of 256 metric 4-spaces (4-maps) form a single 256-page "atlas" with a bonding point at point O, with a total number of mathematical measurements 256 × 4 = 1024.

The sum of all 256 4-metrics (10) intersecting at the point *O* is equal to zero

$$\sum_{k=1}^{256} ds^{(k)2} = \sum_{a=1}^{16} \sum_{b=1}^{16} e_i^{(a)} e_j^{(b)} dx^{i(a)} dx^{j(b)} = 0, \qquad (12)$$

where $k$ = 1,2,3,…,256 corresponds to one of 256 combinations $a,b$.

It is easy to verify that sum (12) is equal to zero, since among 256 × 4 = 1024 signs of all 256 signatures there are 512 {+} and 512 {–}. Thus, Ex. (12) satisfies the "vacuum balance" condition.

### 2.2 Four types of rank multiplication and division rules for different types of $\lambda_{m,n}$-vacuums

Within the framework of the Algebra of Signatures, multiplication and division of signs in the numerators of ranks can be performed according to the following four types of arithmetic rules, which are assigned to four types of metric $\lambda_{m,n}$-vacuums:

I - rules for commutative metric $\lambda_{m,n}$-vacuum (or $\lambda^{I}_{m,n}$-vacuum):

{+} × {+} = {+} {–} × {+} = {–} (13)

{+} × {–} = {–} {–} × {–} = {+}

{+} : {+} = {+} {–} : {+} = {–} (14)

{+} : {–} = {–} {–} : {–} = {+};

H - rules for non-commutative metric $\lambda_{m,n}$-vacuum (or $\lambda^{H}_{m,n}$-vacuum):

{+} × {+} = {+} {–} × {+} = {+} (15)

{+} × {–} = {–} {–} × {–} = {+}

{+} : {+} = {+} {–} : {+} = {+} (16)

{+} : {–} = {–} {–} : {–} = {+};

V - rules for the commutative metric $\lambda_{m,n}$-antivacuum (or $\lambda^{V}_{m,n}$-vacuum):

{+} × {+} = {–} {–} × {+} = {+} (17)

{+} × {–} = {+} {–} × {–} = {–}

{+} : {+} = {–} {–} : {+} = {+} (18)

{+} : {–} = {+} {–} : {–} = {–};

H' - rules for non-commutative metric $\lambda_{m,n}$-antivacuum (or $\lambda^{H'}_{m,n}$-vacuum):

{+} × {+} = {–} {–} × {+} = {–} (19)

{+} × {–} = {+} {–} × {–} = {–}

{+} : {+} = {–} {–} : {+} = {–} (20)

{+} : {–} = {+} {–} : {–} = {–}.

For example, let's write the ranking (8) and several other rankings from the list (11) for four types of $\lambda_{m,n}$-vacuums with the corresponding multiplication rules (13), (15), (17), (19)

$$\begin{array}{cccc} \{+ + + +\} & \{+ + + +\} & \{+ + + +\} & \{+ + + +\} \\ \underline{\{+ + + -\}} & \underline{\{+ + + -\}} & \underline{\{+ + + -\}} & \underline{\{+ + + -\}} \\ (+ + + -)_{I\times} & (+ + + -)_{H\times} & (- - - +)_{V\times} & (- - - +)_{H'\times} \end{array} \qquad (21)$$

$$\begin{array}{cccc} \{- + - +\} & \{- + - +\} & \{- + - +\} & \{- + - +\} \\ \underline{\{- - + -\}} & \underline{\{- - + -\}} & \underline{\{- - + -\}} & \underline{\{- - + -\}} \\ (+ - - -)_{I\times} & (+ - + -)_{H\times} & (- + + +)_{V\times} & (- + - +)_{H'\times} \end{array} \qquad (22)$$

$$\begin{array}{cccc} \{- + - -\} & \{- + - -\} & \{- + - -\} & \{- + - -\} \\ \underline{\{- + + -\}} & \underline{\{- + + -\}} & \underline{\{- + + -\}} & \underline{\{- + + -\}} \\ (+ + - +)_{I\times} & (+ + + +)_{H\times} & (- - + -)_{V\times} & (- - - -)_{H'\times} \end{array} \qquad (23)$$

$$\begin{array}{cccc} \{+ + - +\} & \{+ + - +\} & \{+ + - +\} & \{+ + - +\} \\ \underline{\{- - + +\}} & \underline{\{- - + +\}} & \underline{\{- - + +\}} & \underline{\{- - + +\}} \\ (- - - +)_{I\times} & (- - + +)_{H\times} & (+ + + -)_{V\times} & (+ + - -)_{H'\times} \end{array} \qquad (24)$$

In this case, the sum of signs in the denominators of each quadruple of ranks (21) – (24) is equal to zero, for example, for four ranks (21) we have

$$(+ + + -) \ + \ (+ + + -) \ + \ (- - - +) \ + \ (- - - +) = 0, \qquad (25)$$

and the sum of these signatures is equal to the zero signature

$$(+ + + -) \ + \ (+ + + -) \ + \ (- - - +) \ + \ (- - - +) = (0\ 0\ 0\ 0). \qquad (26)$$

This corresponds to the "vacuum balance" condition.

Taking into account the four rules for multiplication of signs (13), (15), (17), (19), it turns out that at the point $O$ under study (see Figure 1) four $\lambda^{L}{}_{m,n}$-vacuums or 256 × 4 = 1024 metric spaces intersect, which are characterized by metrics (that is, quadratic forms) $ds^{(l)2}$ with the corresponding signatures.

The sum of all four metric $\lambda^{L}{}_{m,n}$-vacuums and, accordingly, the sum of all 1024 metrics $ds^{(l)2}$ is still equal to zero

$$\lambda^{I}{}_{m,n}\text{-vacuum} + \lambda^{H}{}_{m,n}\text{-vacuum} + \lambda^{V}{}_{m,n}\text{-vacuum} + \lambda^{H'}{}_{m,n}\text{-vacuum} = 0, \qquad (27)$$

$$\sum_{l=1}^{1024} ds^{(k)2} = 0, \qquad (28)$$

which satisfies the requirement of maintaining the "vacuum balance". The sum of metric $\lambda^{L}{}_{m,n}$-vacuums (27) {or quadratic forms (28)} will also be called "deep zero".

Metric $\lambda^{L}{}_{m,n}$-vacuums (27) are "supports" for each other and provide complete balancing of the metric emptiness. In what follows, each metric $\lambda^{L}{}_{m,n}$-vacuum will be assigned a specific factorial of zero corresponding to one of the multiplication rules (13), (15), (17), (19):

$$0_I! = 1, \qquad 0_H! = -1, \qquad 0_V! = i, \qquad 0_{H'}! = -i. \qquad (29)$$

so that the sum of these factorials corresponds to "true zero"

$$0_I! + 0_H! + 0_V! + 0_{H'}! = 1 + (-1) + i + (-i) = 0. \qquad (30)$$

The identity of "deep zero" and "true zero" will lead to closed completeness of the developed theory.

### 2.3 Signature matrix

As shown above, the scalar multiplication of the sixteen 4-bases shown in Figure 2, with each other led to the formation of an atlas of 16 × 16 = 256 metric spaces with metrics (10) $ds^{(ab)2} = \mathbf{e}_i^{(a)}\mathbf{e}_j^{(b)}dx^{i(a)}dx^{j(b)}$ with the corresponding signatures. However, there are only 16 different signatures, since there is a 16-fold degeneracy. For example, 16 scalar products of 4-bases shown in Figure 4 result in sixteen quadratic forms (i.e., metrics) with the same signature (– + – +):

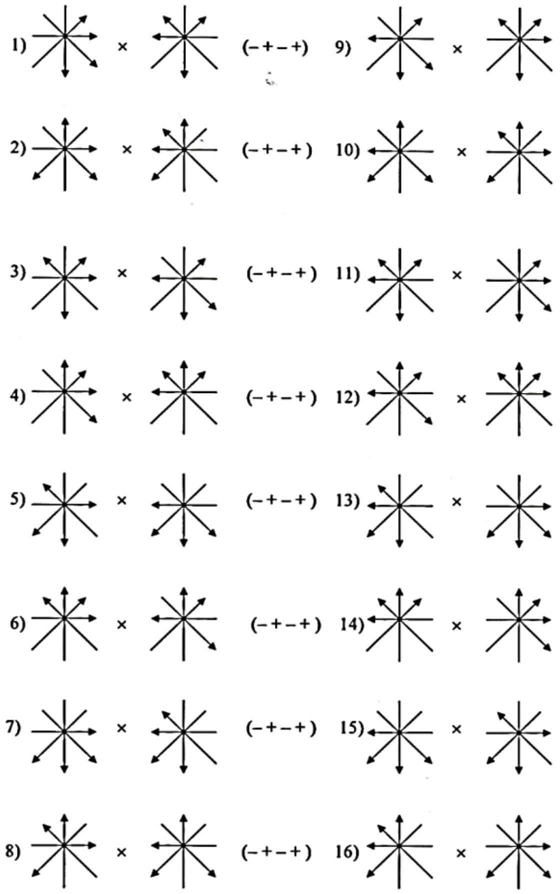


**Figure 4.** Sixteen scalar products of 4-bases, resulting in to metrics with the same signature (– + – +).

Similarly, we obtain 16-fold degeneracy with all other metric spaces. Thus, it is possible to single out only 256 : 16 = 16 types of metric 4-spaces with quadratic forms (i.e., metrics)

$$
\begin{array}{ll}
ds^{(++++)2} = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 & ds^{(----)2} = -dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \\
ds^{(---+)2} = -dx_0^2 - dx_1^2 - dx_2^2 + dx_3^2 & ds^{(+++-)2} = dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2 \\
ds^{(+--+)2} = dx_0^2 - dx_1^2 - dx_2^2 + dx_3^2 & ds^{(-++-)2} = -dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2 \\
ds^{(+---)2} = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 & ds^{(-+++)2} = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \\
ds^{(--+-)2} = -dx_0^2 - dx_1^2 + dx_2^2 - dx_3^2 & ds^{(++-+)2} = dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2 \\
ds^{(-+--)2} = -dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 & ds^{(+-++)2} = dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2 \\
ds^{(+-+-)2} = dx_0^2 - dx_1^2 + dx_2^2 - dx_3^2 & ds^{(-+-+)2} = -dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2 \\
ds^{(++--)2} = dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 & ds^{(--++)2} = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2
\end{array}
\qquad (31)
$$

with the corresponding signatures, which form a matrix

$$
stign\left(ds^{(a,b)2}\right) = \begin{pmatrix} (++++) & (+++-) & (-++-) & (++-+) \\ (---+) & (-+++) & (--++) & (-+-+) \\ (+--+) & (++--) & (+---) & (+-++) \\ (--+-) & (+-+-) & (-+--) & (----) \end{pmatrix}. \qquad (32)
$$

The elements of the matrix of signatures (32) completely coincide with the elements of the matrix of signatures (2) {or (3) in the article [1]}. Therefore, the properties of the signature matrix (32) largely repeat the properties of the signature matrix (see [1]) in the next branch of the theory development.

**2.4 Relationship between signature and 4-space topology**

According to Felix Klein's classification [3], metric spaces with metrics (31) can be divided into three topological types:

**1st type:** 4-spaces whose signatures consist of four identical signs [3]:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 \qquad (+ + + +) \qquad (33)$$
$$- x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \qquad (- - - -)$$

are the so-called null metric 4-spaces. These "spaces" have only one real point, located at the beginning of the light cone. All other points of these extensions are imaginary. In fact, the first of the Exs. (33) describes not the "extent", but a single point (or "white" point), and the second describes the only anti-point (or "black" point).

**2nd type:** 4-spaces whose signatures consist of two positive and two negative signs [3]:

$$x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0 \qquad (+ - - +) \qquad (34)$$
$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0 \qquad (+ + - -)$$
$$x_0^2 - x_1^2 + x_2^2 - x_3^2 = 0 \qquad (+ - + -)$$
$$- x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0 \qquad (- + + -)$$
$$- x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0 \qquad (- - + +)$$
$$- x_0^2 + x_1^2 - x_2^2 + x_3^2 = 0 \qquad (- + - +)$$

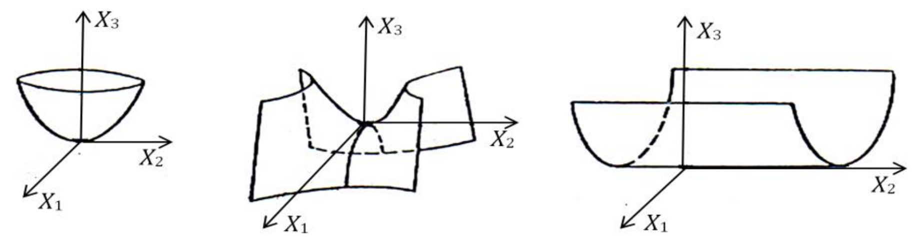
are different variants of 4-dimensional tori.

**3rd type:** 4-spaces whose signatures consist of three identical signs and one opposite one [3]:

$$- x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0 \qquad (- - - +) \qquad (35)$$
$$- x_0^2 - x_1^2 + x_2^2 - x_3^2 = 0 \qquad (- - + -)$$
$$- x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0 \qquad (- + - -)$$
$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \qquad (+ - - -)$$
$$x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0 \qquad (+ + + -)$$
$$x_0^2 + x_1^2 - x_2^2 + x_3^2 = 0 \qquad (+ + - +)$$
$$x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0 \qquad (+ - + +)$$
$$- x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 \qquad (- + + +)$$

are oval 4-surfaces: ellipsoids, elliptic paraboloids, two-sheeted hyperboloids.

A simplified illustration of the connection between the signature of a 2-dimensional space and its topology is shown in Figure 5. This figure shows that the signature of the quadratic form is uniquely related to the topology of the 2-dimensional extent. But not vice versa, the extension topology is a much more capacious concept than the signature of its metric.




a) signature (+ +) $x_3 = x_1^2 + x_2^2$ parabolic surface;
b) signature (– +) $x_3 = x_2^2 - x_1^2$ saddle surface;
c) signature (+ 0) $x_3 = x_1^2$ U-shaped surface

**Figure 5.** Illustration of the connection between the signature of a 2-dimensional space and its topology [3].

### 2.5 Splitting the metric zero

The sum of all 16 metrics (31) is zero:

$$\begin{aligned} ds_\Sigma^2 = {} & ds^{(+---)2} + ds^{(++++)2} + ds^{(---+)2} + ds^{(+--+)2} + \\ & + ds^{(--+-)2} + ds^{(++--)2} + ds^{(-+--)2} + ds^{(+-+-)2} + \\ & + ds^{(-+++)2} + ds^{(----)2} + ds^{(+++-)2} + ds^{(-++-)2} + \\ & + ds^{(++-+)2} + ds^{(--++)2} + ds^{(+-++)2} + ds^{(-+-+)2} = 0. \end{aligned} \tag{36}$$

Indeed, summing metrics (31), we obtain

$$\begin{aligned} ds_\Sigma^2 = {} & (dx_0dx_0 - dx_1dx_1 - dx_2dx_2 - dx_3dx_3) + (dx_0dx_0 + dx_1dx_1 + dx_2dx_2 + dx_3dx_3) + \\ & + (-dx_0dx_0 - dx_1dx_1 + dx_2dx_2 - dx_3dx_3) + (dx_0dx_0 - dx_1dx_1 - dx_2dx_2 + dx_3dx_3) + \\ & + (-dx_0dx_0 - dx_1dx_1 + dx_2dx_2 - dx_3dx_3) + (dx_0dx_0 + dx_1dx_1 - dx_2dx_2 - dx_3dx_3) + \\ & + (-dx_0dx_0 + dx_1dx_1 - dx_2dx_2 - dx_3dx_3) + (dx_0dx_0 - dx_1dx_1 + dx_2dx_2 - dx_3dx_3) + \\ & + (-dx_0dx_0 + dx_1dx_1 + dx_2dx_2 + dx_3dx_3) + (-dx_0dx_0 - dx_1dx_1 - dx_2dx_2 - dx_3dx_3) + \\ & + (dx_0dx_0 + dx_1dx_1 + dx_2dx_2 - dx_3dx_3) + (-dx_0dx_0 + dx_1dx_1 + dx_2dx_2 - dx_3dx_3) + \\ & + (dx_0dx_0 + dx_1dx_1 - dx_2dx_2 + dx_3dx_3) + (-dx_0dx_0 - dx_1dx_1 + dx_2dx_2 + dx_3dx_3) + \\ & + (dx_0dx_0 - dx_1dx_1 + dx_2dx_2 + dx_3dx_3) + (-dx_0dx_0 + dx_1dx_1 - dx_2dx_2 + dx_3dx_3) = 0. \end{aligned} \tag{37}$$

Instead of summing homogeneous terms in Ex. (37), only the signs in front of these terms can be summed. Therefore, the total metric (37) can be represented as a ranking expression:

$$\begin{array}{lcccc} 0 = & \underline{(0\ \ 0\ \ 0\ \ 0)} & + & \underline{(0\ \ 0\ \ 0\ \ 0)} & = 0 \\ 0 = & (+\ +\ +\ +) & + & (-\ -\ -\ -) & = 0 \\ 0 = & (-\ -\ -\ +) & + & (+\ +\ +\ -) & = 0 \\ 0 = & (+\ -\ -\ +) & + & (-\ +\ +\ -) & = 0 \\ 0 = & (-\ -\ +\ -) & + & (+\ +\ -\ +) & = 0 \\ 0 = & (+\ +\ -\ -) & + & (-\ -\ +\ +) & = 0 \\ 0 = & (-\ +\ -\ -) & + & (+\ -\ +\ +) & = 0 \\ 0 = & (+\ -\ +\ -) & + & (-\ +\ -\ +) & = 0 \\ 0 = & \underline{(-\ +\ +\ +)} & + & \underline{(+\ -\ -\ -)} & = 0 \\ 0 = & (0\ \ 0\ \ 0\ \ 0) & + & (0\ \ 0\ \ 0\ \ 0)_{+} & = 0, \\ & {}_{+} & & & \end{array} \tag{38}$$

where the summation (or subtraction) of signs is carried out according to the rules:

$$(+) + (+) = 2(+), \quad (-) + (+) = (0), \quad (+) - (+) = (0), \quad (-) - (+) = 2(-), \qquad (39)$$
$$(+) + (-) = (0), \quad (-) + (-) = 2(-), \quad (+) - (-) = 2(+), \quad (-) - (-) = (0).$$

The sum of the signs, both in the columns of the ranks (38) and in their lines between the ranks, is equal to zero. Therefore, this ranking identity will be called the "splitting of the metric zero".

**2.6 Operations with ranks**

The ranking expression (38) makes it possible to perform some operations in the vicinity of the investigated point *O* (see Figure 1) without violating the “vacuum balance”. Such operations include, for example, the symmetrical transfer of the first and last columns to the other side of equality with sign inversion, while observing line-by-line and column-by-column vacuum balance:

$$\begin{array}{lcccccc}
0 = & \underline{(0\ \ 0\ \ 0)} & + & \underline{(0\ \ 0\ \ 0)} & = 0 & & \qquad (40) \\
- = & (+\ \ +\ \ +) & + & (-\ \ -\ \ -) & = + & \to & 0 \\
+ = & (-\ \ -\ \ +) & + & (+\ \ +\ \ -) & = - & \to & 0 \\
- = & (-\ \ -\ \ +) & + & (+\ \ +\ \ -) & = + & \to & 0 \\
+ = & (-\ \ +\ \ -) & + & (+\ \ -\ \ +) & = - & \to & 0 \\
- = & (+\ \ -\ \ -) & + & (-\ \ +\ \ +) & = + & \to & 0 \\
+ = & (+\ \ -\ \ -) & + & (-\ \ +\ \ +) & = - & \to & 0 \\
- = & (-\ \ +\ \ -) & + & (+\ \ -\ \ +) & = + & \to & 0 \\
+ = & \underline{(+\ \ +\ \ +)} & + & \underline{(-\ \ -\ \ -)} & = - & \to & 0 \\
0 = & (0\ \ 0\ \ 0)_+ & + & (0\ \ 0\ \ 0)_+ & = 0. & &
\end{array}$$

Similarly, any columns of the rank expression (38) can be symmetrically transferred to the other side like (40).

It is possible to transfer any string from the numerators of the rankings (38) to their denominators, also with the inversion of signs, and observing the line-by-line vacuum balance, for example:

$$\begin{array}{lcccc}
0 = & (+\ \ +\ \ +\ \ +) & + & (-\ \ -\ \ -\ \ -) & = 0 \qquad (41) \\
0 = & (-\ \ -\ \ -\ \ +) & + & (+\ \ +\ \ +\ \ -) & = 0 \\
0 = & (+\ \ -\ \ -\ \ +) & + & (-\ \ +\ \ +\ \ -) & = 0 \\
0 = & (+\ \ +\ \ -\ \ -) & + & (-\ \ -\ \ +\ \ +) & = 0 \\
0 = & (-\ \ +\ \ -\ \ -) & + & (+\ \ -\ \ +\ \ +) & = 0 \\
0 = & (+\ \ -\ \ +\ \ -) & + & (-\ \ +\ \ -\ \ +) & = 0 \\
0 = & \underline{(-\ \ +\ \ +\ \ +)} & + & \underline{(+\ \ -\ \ -\ \ -)} & = 0 \\
0 = & (+\ \ +\ \ -\ \ +)_+ & + & (-\ \ -\ \ +\ \ -)_+ & = 0.
\end{array}$$

Mixed line and column transfer operations are also possible, which do not violate the conditions of line-by-line vacuum balance, for example

$$
\begin{array}{lcccccc}
- = & (+\ \ +\ \ +) & + & (-\ \ -\ \ -) & = + & \rightarrow 0 \\
- = & (-\ \ -\ \ -) & + & (+\ \ +\ \ +) & = + & \rightarrow 0 \\
- = & (+\ \ -\ \ -) & + & (-\ \ +\ \ +) & = + & \rightarrow 0 \\
+ = & (+\ \ +\ \ -) & + & (-\ \ -\ \ +) & = - & \rightarrow 0 \\
+ = & (-\ \ +\ \ -) & + & (+\ \ -\ \ +) & = - & \rightarrow 0 \\
+ = & (+\ \ -\ \ +) & + & (-\ \ +\ \ -) & = - & \rightarrow 0 \\
\underline{- =} & \underline{(-\ \ +\ \ +)} & + & \underline{(+\ \ -\ \ -)} & \underline{= +} & \underline{\rightarrow 0} \\
- = & (+\ \ +\ \ -)_+ & + & (-\ \ -\ \ +)_+ & = + & \rightarrow 0.
\end{array}
\qquad (42)
$$

Such a ranking operations correspond to certain vacuum symmetries, which will be considered in the following articles of the proposed project.

**2.7 Bilateral metric space**

We transfer the signatures (– + + +) and (+ – – –) from the numerators of the ranks (38) to their denominators

$$
\begin{array}{lclc}
(+\ \ +\ \ +\ \ +) & + & (-\ \ -\ \ -\ \ -) & = 0 \\
(-\ \ -\ \ -\ \ +) & + & (+\ \ +\ \ +\ \ -) & = 0 \\
(+\ \ -\ \ -\ \ +) & + & (-\ \ +\ \ +\ \ -) & = 0 \\
(-\ \ -\ \ +\ \ -) & + & (+\ \ +\ \ -\ \ +) & = 0 \\
(+\ \ +\ \ -\ \ -) & + & (-\ \ -\ \ +\ \ +) & = 0 \\
(-\ \ +\ \ -\ \ -) & + & (+\ \ -\ \ +\ \ +) & = 0 \\
\underline{(+\ \ -\ \ +\ \ -)} & + & \underline{(-\ \ +\ \ -\ \ +)} & = 0 \\
(+\ \ -\ \ -\ \ -)_+ & + & (-\ \ +\ \ +\ \ +)_+ & = 0.
\end{array}
\qquad (43)
$$

In expanded form, the ranks (43) have the following form

$$
\begin{array}{ll}
ds^{(++++)2} = \ dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 & ds^{(----)2} = -\,dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \\
ds^{(---+)2} = -\,dx_0^2 - dx_1^2 - dx_2^2 + dx_3^2 & ds^{(+++-)2} = \ dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2 \\
ds^{(+--+)2} = \ dx_0^2 - dx_1^2 - dx_2^2 + dx_3^2 & ds^{(-++-)2} = -\,dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2 \\
ds^{(--+-)2} = -\,dx_0^2 - dx_1^2 + dx_2^2 - dx_3^2 & ds^{(++-+)2} = \ dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2 \\
ds^{(-+--)2} = -\,dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 & ds^{(+-++)2} = \ dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2 \\
ds^{(+-+-)2} = \ dx_0^2 - dx_1^2 + dx_2^2 - dx_3^2 & ds^{(-+-+)2} = -\,dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2 \\
\underline{ds^{(++--)2} = \ dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2} & \underline{ds^{(--++)2} = -\,dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2} \\
ds^{(+---)2} = \ dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 & ds^{(-+++)2} = -\,dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.
\end{array}
\qquad (44)
$$

The ranking expression (44) is equivalent to the fact that the addition (i.e., additive overlay) of 7-metric spaces with signatures (topologies) indicated in the numerator of the left ranking (43) form a metric Minkowski 4-space with the metric

$$ds^{(+---)2} = c^2dt^2 - dx^2 - dy^2 - dz^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \quad \text{with signature } (+\,-\,-\,-), \qquad (44)$$

where $$ds^{(+---)2} = ds^{(++++)2} + ds^{(---+)2} + ds^{(+--+)2} + ds^{(--+-)2} + ds^{(++--)2} + ds^{(-+--)2} + ds^{(+-+-)2}, \qquad (45)$$

this Minkowski 4-space will be conditionally called the outer side of the $\lambda_{m,n}$-vacuum (or subcont – short for "substantial continuum").

In this case, the additive imposition of 7 metric spaces with signatures indicated in the numerator of the right-th rank (43) forms a metric Minkowski 4-antispace with the metric

$$ds^{(-+++)2} = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \quad \text{with signature } (-+++), \tag{46}$$

where
$$ds^{(-+++)2} = ds^{(----)2} + ds^{(+++-)2} + ds^{(-++-)2} + ds^{(++-+)2} + ds^{(--++)2} + ds^{(+-++)2} + ds^{(-+-+)2}. \tag{47}$$

This metric Minkowski 4-antispace will be conditionally called the inner side of the $\lambda_{m,n}$-vacuum (or antisubcont – short for "antisubstantial continuum".

The concepts of "**subcont**" and "**antisubcont**" are mental constructions that are intended only to create the illusion of "visibility" of two adjacent mutually opposite sides of one $\lambda_{m,n}$-vacuum. If one side of a sheet of paper is painted blue, and the other side of the same sheet is painted red, then the blue side of the sheet can be associated with the "subcont", and its red side with the "antisubcont". The concepts of "subcont" and "antisubcont" are introduced only to facilitate the visualization of intra-vacuum processes, but they have nothing to do with reality. However, as will be shown in the following articles of this project, using these mental concepts it is possible to inspire real vacuum effects.

The operation described by the ranking expression (43) allows you to mentally "reveal" from the void the two-sided $\lambda_{m,n}$-vacuum with the number of mathematical dimensions 4 + 4 = 8 = $2^3$. We propose to call such a two-sided 8-dimensional space $2^3$-$\lambda_{m,n}$-vacuum, provided that the $2^3$-$\lambda_{m,n}$-vacuum balance is maintained

$$ds^{(+---)2} + ds^{(-+++)2} = 0, \tag{48}$$

with ranking equivalent (+ – – –) + (– + + +) = (0 0 0 0), or in transposed form

$$\begin{array}{c} (+\,-\,-\,-) \\ \underline{(-\,+\,+\,+)} \\ (0\;0\;0\;0)_{+} \end{array} \tag{49}$$

In the terminology proposed here, the ranking expression (38) is equivalent to the balance condition for a $2^6$-$\lambda_{m,n}$-vacuum with 4-dimensional sides (or faces), since the number of mathematical dimensions of such a 16-faced extension:

$$4 \times 16 = 64 = 2^6. \tag{50}$$

Philosophical understanding of the ranking expression (38) can lead to the roots of religious and mythological traditions, where the number 7 has the sacred meaning of "Seven Heavens", and two mutually opposite sides of the $2^3$-$\lambda_{m,n}$-vacuum corresponds to the perception of reality through ascending logic to the Hegelian dialectic.

Here, for the first time, mathematical (speculative) calculations of the Algebra of Signature led to the following very important practical conclusion. The vacuum balance condition led to the need to assume that the empty extent surrounding us has at least sixteen 4-dimensional "faces" with signatures (32). At the same time, in some cases, the number of faces of such an empty extent can be reduced to two with signatures (+ – – –) and (– + + +), and in a number of other problems it can be increased to infinity (see section 9).

In other words, it is necessary to realize that the space around us has at least two sides: "external" and "internal", which can be conditionally called "subcont" and "antisubcont". This will require a full review of our speculative attitude to reality, but as it turns out below, one-sided theories inevitably lead to unsolvable paradoxes, and 16-sided (or at least two-sided) theories allow us to significantly expand the range of tasks to be solved.

Recall that in A. Einstein's General Relativity there is only one metric 4-space with a signature, for example, (+ – – –). Whereas in the geometrized vacuum physics developed here, based on the Algebra of Signatures, any $\lambda_{m,n}$-vacuum can have at least two sides (i.e. mutually opposite metric 4-spaces): the outer side (or subcont) with signatures (+ – – –) and the inner side (or antisubcontent) with the signature (– + + +).

### 2.8 Binary triads

Not only the ranking expression (38) leads to the antipodal dyad: "4-space - 4-antispace" Minkowski with opposite signatures (+ – – –) and (– + + +). The following ranking expressions also lead to this dyad:

These ranking expressions (binary triads) also satisfy the vacuum balance condition and play an important role in "vacuum chromodynamics", which will be described in the following articles of this project.

$$\begin{array}{lcll} (-\ -\ -\ +) & + & (+\ +\ +\ -) & = 0 \\ (+\ -\ +\ -) & + & (-\ +\ -\ +) & = 0 \\ \underline{(+\ +\ -\ -)} & + & \underline{(-\ -\ +\ +)} & = 0 \\ (+\ -\ -\ -)_+ & + & (-\ +\ +\ +)_+ & = 0, \end{array} \qquad (51)$$

$$\begin{array}{lcll} (-\ -\ +\ -) & + & (+\ +\ -\ +) & = 0 \\ (+\ +\ -\ -) & + & (-\ -\ +\ +) & = 0 \\ \underline{(+\ -\ -\ +)} & + & \underline{(-\ +\ +\ -)} & = 0 \\ (+\ -\ -\ -)_+ & + & (-\ +\ +\ +)_+ & = 0, \end{array} \qquad (52)$$

$$\begin{array}{lcll} (-\ -\ +\ -) & + & (+\ +\ -\ +) & = 0 \\ (+\ +\ -\ -) & + & (-\ -\ +\ +) & = 0 \\ \underline{(+\ -\ -\ +)} & + & \underline{(-\ +\ +\ -)} & = 0 \\ (+\ -\ -\ -)_+ & + & (-\ +\ +\ +)_+ & = 0. \end{array} \qquad (53)$$

### 2.9 Transverse bundle of $\lambda_{m,n}$-vacuum

Like the ranking expression (41) and (43), any pair of metric 4-spaces with mutually opposite signatures can be represented as a sum of 7 + 7 = 14 metric extensions with other signatures.

For example, the conjugate pair of metrics $ds^{(-++-)2}$ and $ds^{(+--+)2}$ with mutually opposite signatures (– + + –) and (+ – – +) can be expressed by summing (i.e., additive superposition) 7 + 7 = 14 metric 4-spaces with signatures

$$\begin{array}{lcll} -(+\ +\ +\ +) & + & (-\ -\ -\ -) & = 0 \\ (-\ -\ -\ +) & + & (+\ +\ +\ -) & = 0 \\ (-\ -\ +\ -) & + & (+\ +\ -\ +) & = 0 \\ (+\ +\ -\ -) & + & (-\ -\ +\ +) & = 0 \\ (-\ +\ -\ -) & + & (+\ -\ +\ +) & = 0 \\ (+\ -\ +\ -) & + & (-\ +\ -\ +) & = 0 \\ \underline{(-\ +\ +\ +)} & + & \underline{(+\ -\ -\ -)} & = 0 \\ (-\ +\ +\ -)_+ & + & (+\ -\ -\ +)_+ & = 0. \end{array} \qquad (54)$$

Similarly, out of 256 metrics with signatures (11), 256 : 2 = 128 conjugate pairs of metrics can be distinguished, each of which can be expressed in terms of an additive superposition of 7 + 7 = 14 metric 4-subspaces with corresponding signatures while maintaining a vacuum balance.

In turn, the conjugate pairs of 4-subspaces can be similarly decomposed into sums of 7 + 7 = 14 subspaces, and this can continue indefinitely.

It turns out that the light-geometry of the void is balanced with respect to zero, in which the "vacuum" is first represented as an infinite number of $\lambda_{m,n}$-vacuums nested into each other (see § 1 and Figure 2 in the article [1]). This representation of emptiness is called the longitudinal stratification (bundle) of "vacuum". Then each $\lambda_{m,n}$-vacuum splits into an infinite number of metric 4-subspaces, 4-sub-subspaces, and so on. with 16 types of signatures (or topologies, see § 4) without violating the vacuum balance. Such an infinite splitting of each $\lambda_{m,n}$-vacuum will be called the transverse bundle of the "vacuum".

The longitudinal and transverse stratification (bundle) of the "vacuum" leads to the fact that at each point of the void (including the point $O$ under study, see Figure 1) there is an additive imposition of an infinite number of metric 4-spaces with 16 types of signatures (i.e., topologies, among which are 6 types of tori (34) and 8 types of oval surfaces (35)), which completely compensate for each other's manifestations (i.e. the condition of "vacuum balance" is observed). This leads to the formation of a zero Ricci-flat space, which is in many ways similar to a compact Calabi-Yau manifold (i.e., a multidimensional complex torus) (see Figure 6).

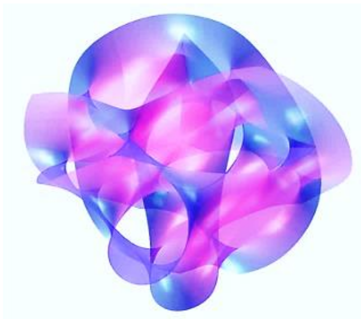


**Figure 6.** One of the implementations of a 2D projection of a 3D visualization of a local area of a 10-dimensional Calabi-Yau manifold [4].

### 2.10 Spin-tensor representation of metrics with different signatures

Let's consider the metric

$$ds^{(+---)2} = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \text{ with signature } (+---). \quad (55)$$

For brevity, we omit the signs of the differentials in the metric (55)

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2. \quad (56)$$

As is known, the quadratic form (56) is a determinant of the Hermitian 2×2-matrix

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}_{det} = \begin{vmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{vmatrix} = x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \text{ with signature } (+---). \quad (57)$$

It is easy to verify that this matrix is Hermitian by direct calculation

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}^+ = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}. \quad (58)$$

In the theory of spinors, matrices of the form (58) are called second-rank mixed Hermitian spin-tensors [5].

Let's represent 2×2-matrix (58) in expanded form

$$\mathrm{A}_4 = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} = x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - x_1 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - x_3 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (59)$$

where $\sigma_0^{(+---)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\sigma_1^{(+---)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$; $\sigma_2^{(+---)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_3^{(+---)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a set of Pauli matrices.

In the theory of spinors, $A_4$-matrices of the form (59) are assigned one-to-one correspondence with quaternions of the type

$$q = x_0 + \vec{e}_1 x_1 + \vec{e}_2 x_2 + \vec{e}_3 x_3,$$

under isomorphism

$$\vec{e}_1 \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}; \quad \vec{e}_2 \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \vec{e}_3 \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (60)$$

Similarly, each quadratic form with the corresponding signature (32):

$$ds^{(++++)2} = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \qquad ds^{(----)2} = -dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \qquad (61)$$

$$ds^{(---+)2} = -dx_0^2 - dx_1^2 - dx_2^2 + dx_3^2 \qquad ds^{(+++-)2} = dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2$$

$$ds^{(+--+)2} = dx_0^2 - dx_1^2 - dx_2^2 + dx_3^2 \qquad ds^{(-++-)2} = -dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2$$

$$ds^{(--+-)2} = -dx_0^2 - dx_1^2 + dx_2^2 - dx_3^2 \qquad ds^{(++-+)2} = dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2$$

$$ds^{(-+--)2} = -dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 \qquad ds^{(+-++)2} = dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2$$

$$ds^{(+-+-)2} = dx_0^2 - dx_1^2 + dx_2^2 - dx_3^2 \qquad ds^{(-+-+)2} = -dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2$$

$$ds^{(++--)2} = dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 \qquad ds^{(--++)2} = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2$$

$$ds^{(+---)2} = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \qquad ds^{(-+++)2} = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$

can be represented as a spin-tensor or an $A_4$-matrix, which are shown in Table 1:

**Table 1.** Spintensors and $A_4$-matrices with different signatures.

| | |
|---|---|
| 1 | $\begin{pmatrix} x_0+ix_3 & ix_1+x_2 \\ ix_1-x_2 & x_0-ix_3 \end{pmatrix}_{det} = x_0^2+x_1^2+x_2^2+x_3^2 = 0,$ signature $(++++)$.<br>$\begin{pmatrix} x_0+ix_3 & ix_1+x_2 \\ ix_1-x_2 & x_0-ix_3 \end{pmatrix} = x_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + x_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x_3\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};$<br>where<br>$\sigma_0^{(++++)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \sigma_1^{(++++)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};\ \sigma_2^{(++++)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \sigma_3^{(++++)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$ |
| 2 | $\begin{pmatrix} x_0+x_3 & ix_1+x_2 \\ ix_1-x_2 & x_0-x_3 \end{pmatrix}_{det} = x_0^2+x_1^2+x_2^2-x_3^2 = 0,$ signature $(+++-)$.<br>$\begin{pmatrix} x_0+x_3 & ix_1+x_2 \\ ix_1-x_2 & x_0-x_3 \end{pmatrix} = x_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + x_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - x_3\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$<br>where<br>$\sigma_0^{(+++-)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \sigma_1^{(+++-)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};\ \sigma_2^{(+++-)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \sigma_3^{(+++-)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$ |
| 3 | $\begin{pmatrix} x_0+ix_3 & ix_1+x_2 \\ ix_1-x_2 & -x_0+ix_3 \end{pmatrix}_{det} = -x_0^2+x_1^2+x_2^2-x_3^2 = 0,$ signature $(-++-)$.<br>$\begin{pmatrix} x_0+ix_3 & ix_1+x_2 \\ ix_1-x_2 & -x_0+ix_3 \end{pmatrix} = -x_0\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + x_1\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + x_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - x_3\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix};$<br>where<br>$\sigma_0^{(-++-)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};\ \sigma_1^{(-++-)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};\ \sigma_2^{(-++-)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \sigma_3^{(-++-)} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}.$ |

4

$$\begin{pmatrix} x_0 + ix_3 & x_1 + x_2 \\ -x_1 + x_2 & x_0 - ix_3 \end{pmatrix}_{det} = x_0^2 + x_1^2 - x_2^2 + x_3^2 = 0, \quad \text{signature } (+ + - +).$$

$$\begin{pmatrix} x_0 + ix_3 & x_1 + x_2 \\ -x_1 + x_2 & x_0 - ix_3 \end{pmatrix} = x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};$$

where

$$\sigma_0^{(++-+)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \sigma_1^{(++-+)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \sigma_2^{(++-+)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix};\ \sigma_3^{(++-+)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

5

$$\begin{pmatrix} x_0 + x_3 & ix_1 + x_2 \\ -ix_1 + x_2 & -x_0 + x_3 \end{pmatrix}_{det} = -x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0, \quad \text{signature } (- - - +).$$

$$\begin{pmatrix} x_0 + x_3 & ix_1 + x_2 \\ -ix_1 + x_2 & -x_0 + x_3 \end{pmatrix} = -x_0 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - x_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

where

$$\sigma_0^{(---+)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1^{(---+)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \sigma_2^{(---+)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ \sigma_3^{(---+)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

6

$$\begin{pmatrix} x_0 + x_3 & ix_1 + x_2 \\ ix_1 - x_2 & -x_0 + x_3 \end{pmatrix}_{det} = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \quad \text{signature } (- + + +).$$

$$\begin{pmatrix} x_0 + x_3 & ix_1 + x_2 \\ ix_1 - x_2 & -x_0 + x_3 \end{pmatrix} = -x_0 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

where

$$\sigma_0^{(-+++)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1^{(-+++)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ \sigma_2^{(-+++)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \sigma_3^{(-+++)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are the Cayley matrices.

7

$$\begin{pmatrix} x_0 + x_3 & x_1 + x_2 \\ x_1 - x_2 & -x_0 + x_3 \end{pmatrix}_{det} = -x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0, \quad \text{signature } (- - + +).$$

$$\begin{pmatrix} x_0 + x_3 & x_1 + x_2 \\ x_1 - x_2 & -x_0 + x_3 \end{pmatrix} = -x_0 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - x_1 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

where

$$\sigma_0^{(--++)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};\ \sigma_1^{(--++)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix};\ \sigma_2^{(--++)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \sigma_3^{(--++)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

8

$$\begin{pmatrix} x_0 + x_3 & -x_1 + x_2 \\ x_1 + x_2 & -x_0 + x_3 \end{pmatrix}_{det} = -x_0^2 + x_1^2 - x_2^2 + x_3^2 = 0, \quad \text{signature } (- + - +).$$

$$\begin{pmatrix} x_0 + x_3 & -x_1 + x_2 \\ x_1 + x_2 & -x_0 + x_3 \end{pmatrix} = -x_0 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

where

$$\sigma_0^{(-+-+)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1^{(-+-+)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \sigma_2^{(-+-+)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ \sigma_3^{(-+-+)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

9

$$\begin{pmatrix} x_0 - ix_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 + ix_3 \end{pmatrix}_{det} = x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0, \quad \text{signature } (+ - - +).$$

$$\begin{pmatrix} x_0 - ix_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 + ix_3 \end{pmatrix} = x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - x_1 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + x_3 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix};$$

where

$$\sigma_0^{(+--+)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1^{(+--+)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ \sigma_2^{(+--+)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix};\ \sigma_3^{(+--+)} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

$\begin{pmatrix} x_0-x_3 & x_1+x_2 \\ -x_1+x_2 & x_0+x_3 \end{pmatrix}_{det} = x_0^2+x_1^2-x_2^2-x_3^2=0,$ signature $(++--)$.

10 $\begin{pmatrix} x_0-x_3 & x_1+x_2 \\ -x_1+x_2 & x_0+x_3 \end{pmatrix} = x_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + x_1\begin{pmatrix}0&1\\-1&0\end{pmatrix} - x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix} - x_3\begin{pmatrix}1&0\\0&-1\end{pmatrix};$

where

$\sigma_0^{(++--)} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \sigma_1^{(++--)} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \sigma_2^{(++--)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_3^{(++--)} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$

---

$\begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix}_{det} = x_0^2-x_1^2-x_2^2-x_3^2=0,$ signature $(+---)$.

11 $\begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix} = x_0\begin{pmatrix}1&0\\0&1\end{pmatrix} - x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix} - x_2\begin{pmatrix}0&-i\\i&0\end{pmatrix} - x_3\begin{pmatrix}-1&0\\0&1\end{pmatrix};$

where

$\sigma_0^{(+---)} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \sigma_1^{(+---)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_2^{(+---)} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \sigma_3^{(+---)} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}.$

---

$\begin{pmatrix} x_0+ix_3 & x_1+x_2 \\ x_1-x_2 & x_0-ix_3 \end{pmatrix}_{det} = x_0^2-x_1^2+x_2^2+x_3^2=0,$ signature $(\pm + +)$.

12 $\begin{pmatrix} x_0+ix_3 & x_1+x_2 \\ x_1-x_2 & x_0-ix_3 \end{pmatrix} = x_0\begin{pmatrix}1&0\\0&1\end{pmatrix} - x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix} + x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix} + x_3\begin{pmatrix}i&0\\0&-i\end{pmatrix};$

where

$\sigma_0^{(+-++)} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \sigma_1^{(+-++)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_2^{(+-++)} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \sigma_3^{(+-++)} = \begin{pmatrix}i&0\\0&-i\end{pmatrix}.$

---

$\begin{pmatrix} -x_0+ix_3 & x_1+x_2 \\ x_1-x_2 & x_0+ix_3 \end{pmatrix}_{det} = -x_0^2-x_1^2+x_2^2-x_3^2=0,$ signature $(--+-)$.

13 $\begin{pmatrix} -x_0+ix_3 & x_1+x_2 \\ x_1-x_2 & x_0+ix_3 \end{pmatrix} = -x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix} - x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix} + x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix} - x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix};$

where

$\sigma_0^{(--+-)} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \sigma_1^{(--+-)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_2^{(--+-)} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \sigma_3^{(--+-)} = \begin{pmatrix}-i&0\\0&-i\end{pmatrix}.$

---

$\begin{pmatrix} x_0-x_3 & x_1+x_2 \\ x_1-x_2 & x_0+x_3 \end{pmatrix}_{det} = x_0^2-x_1^2+x_2^2-x_3^2=0,$ signature $(+-+-)$.

14 $\begin{pmatrix} x_0-x_3 & x_1+x_2 \\ x_1-x_2 & x_0+x_3 \end{pmatrix} = x_0\begin{pmatrix}1&0\\0&1\end{pmatrix} - x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix} + x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix} - x_3\begin{pmatrix}1&0\\0&-1\end{pmatrix};$

where

$\sigma_0^{(+-+-)} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \sigma_1^{(+-+-)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_2^{(+-+-)} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \sigma_3^{(+-+-)} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$

---

$\begin{pmatrix} -x_0+ix_3 & x_1+x_2 \\ -x_1+x_2 & x_0+ix_3 \end{pmatrix}_{det} = -x_0^2+x_1^2-x_2^2-x_3^2=0,$ signature $(-+--)$.

15 $\begin{pmatrix} -x_0+ix_3 & x_1+x_2 \\ -x_1+x_2 & x_0+ix_3 \end{pmatrix} = -x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix} + x_1\begin{pmatrix}0&1\\-1&0\end{pmatrix} - x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix} - x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix};$

where

$\sigma_0^{(-+--)} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \sigma_1^{(-+--)} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \sigma_2^{(-+--)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_3^{(-+--)} = \begin{pmatrix}-i&0\\0&-i\end{pmatrix}.$

---

$\begin{pmatrix} -x_0+ix_3 & x_1+ix_2 \\ x_1-ix_2 & x_0+ix_3 \end{pmatrix}_{det} = -x_0^2-x_1^2-x_2^2-x_3^2=0,$ signature $(----)$.

16 $\begin{pmatrix} -x_0+ix_3 & x_1+ix_2 \\ x_1-ix_2 & x_0+ix_3 \end{pmatrix} = -x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix} - x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix} - x_2\begin{pmatrix}0&-i\\i&0\end{pmatrix} - x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix};$

where

$\sigma_0^{(----)} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \sigma_1^{(----)} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \sigma_2^{(----)} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \sigma_3^{(----)} = \begin{pmatrix}-i&0\\0&-i\end{pmatrix}.$

Each $A_4$-matrix from the Table 1 is associated with a "colored" quaternion with the corresponding stignature (see Table 2), where following objects are used as imaginary units

$$\vec{e}_1 \to \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{e}_2 \to \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \vec{e}_3 \to \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \vec{e}_4 \to \sigma_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{62}$$

$$\vec{e}_5 \to \sigma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{e}_6 \to \sigma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \vec{e}_7 \to \sigma_7 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \vec{e}_8 \to \sigma_8 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\vec{e}_9 \to \sigma_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{e}_{10} \to \sigma_{10} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \vec{e}_{11} \to \sigma_{11} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \vec{e}_{12} \to \sigma_{12} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\vec{e}_{13} \to \sigma_{13} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{e}_{14} \to \sigma_{14} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \vec{e}_{15} \to \sigma_{15} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \quad \vec{e}_{16} \to \sigma_{16} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

where $\sigma_{ij}$ are the Pauli-Cayley spin-matrices, which are generators of the Clifford algebra and satisfy the conditions

$$\sigma_i\sigma_j + \sigma_j\sigma_i = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{при } i \neq j, \\ 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{при } i = j. \end{cases} \tag{63}$$

In Table 1 shows only particular cases of spin-tensor representations of quadratic forms. For example, the quadratic form $s^{(+---)2} = x_0^2 - x_1^2 - x_2^2 - x_3^2$ is the determinant of all the following 2×2-matrices (Hermitian spin-tensors):

$$\begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix} \begin{pmatrix} x_0-x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0+x_3 \end{pmatrix} \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \begin{pmatrix} x_0-x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0+x_3 \end{pmatrix} \begin{pmatrix} ix_1-x_2 & -x_0+x_3 \\ x_0+x_3 & ix_1+x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_0+x_1 & x_3+ix_2 \\ x_3-ix_2 & x_0-x_1 \end{pmatrix} \begin{pmatrix} x_0-x_1 & x_3-ix_2 \\ x_3+ix_2 & x_0+x_1 \end{pmatrix} \begin{pmatrix} x_0+x_1 & x_3-ix_2 \\ x_3+ix_2 & x_0-x_1 \end{pmatrix} \begin{pmatrix} x_0-x_1 & x_3+ix_2 \\ x_3-ix_2 & x_0+x_1 \end{pmatrix} \begin{pmatrix} ix_2-x_1 & -x_0+x_3 \\ x_0+x_3 & ix_2+x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_0+x_2 & x_1+ix_3 \\ x_1-ix_3 & x_0-x_2 \end{pmatrix} \begin{pmatrix} x_0-x_2 & x_1-ix_3 \\ x_1+ix_3 & x_0+x_2 \end{pmatrix} \begin{pmatrix} x_0+x_2 & x_1-ix_3 \\ x_1+ix_3 & x_0-x_2 \end{pmatrix} \begin{pmatrix} x_0-x_2 & x_1+ix_3 \\ x_1-ix_3 & x_0+x_2 \end{pmatrix} \begin{pmatrix} ix_1-x_3 & -x_0+x_2 \\ x_0+x_2 & ix_1+x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_0+x_3 & x_2+ix_1 \\ x_2-ix_1 & x_0-x_3 \end{pmatrix} \begin{pmatrix} x_0-x_3 & x_2-ix_1 \\ x_2+ix_1 & x_0+x_3 \end{pmatrix} \begin{pmatrix} x_0+x_3 & x_2-ix_1 \\ x_2+ix_1 & x_0-x_3 \end{pmatrix} \begin{pmatrix} x_0-x_3 & x_2+ix_1 \\ x_2-ix_1 & x_0+x_3 \end{pmatrix} \begin{pmatrix} ix_3-x_2 & -x_0+x_1 \\ x_0+x_1 & ix_3+x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_0+x_1 & x_2+ix_3 \\ x_2-ix_3 & x_0-x_1 \end{pmatrix} \begin{pmatrix} x_0-x_1 & x_2-ix_3 \\ x_2+ix_3 & x_0+x_1 \end{pmatrix} \begin{pmatrix} x_0+x_1 & x_2-ix_3 \\ x_2+ix_3 & x_0-x_1 \end{pmatrix} \begin{pmatrix} x_0-x_1 & x_2+ix_3 \\ x_2-ix_3 & x_0+x_1 \end{pmatrix} \begin{pmatrix} ix_2-x_3 & -x_0+x_1 \\ x_0+x_1 & ix_2+x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_0+x_2 & x_3+ix_1 \\ x_3-ix_1 & x_0-x_2 \end{pmatrix} \begin{pmatrix} x_0-x_2 & x_3-ix_1 \\ x_3+ix_1 & x_0+x_2 \end{pmatrix} \begin{pmatrix} x_0+x_2 & x_3-ix_1 \\ x_3+ix_1 & x_0-x_2 \end{pmatrix} \begin{pmatrix} x_0-x_2 & x_3+ix_1 \\ x_3-ix_1 & x_0+x_2 \end{pmatrix} \begin{pmatrix} ix_3-x_1 & -x_0+x_2 \\ x_0+x_2 & ix_3+x_1 \end{pmatrix}$$

$$\begin{pmatrix} ix_2-x_1 & -x_0+x_3 \\ x_0+x_3 & ix_2+x_1 \end{pmatrix} \begin{pmatrix} ix_2-x_1 & x_0+x_3 \\ -x_0+x_3 & ix_2+x_1 \end{pmatrix} \begin{pmatrix} ix_1-x_3 & x_0+x_2 \\ -x_0+x_2 & ix_1+x_3 \end{pmatrix} \begin{pmatrix} ix_2-x_3 & x_0+x_1 \\ -x_0+x_1 & ix_2+x_3 \end{pmatrix} \begin{pmatrix} ix_3-x_1 & x_0+x_2 \\ -x_0+x_2 & ix_3+x_1 \end{pmatrix} \tag{64}$$

**Table 2.** Quadratic forms, $A_4$-matrices and "colored" quaternions.

| Quadratic form | $A_4$-matrix | "Colored" quaternion | Stignatur |
|---|---|---|---|
| $ds_1^2=x_0^2+x_1^2+x_2^2+x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&i\\i&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ | $z_1=x_0+ix_1+jx_2+kx_3$ | {+ + + +} |
| $ds_2^2=x_0^2-x_1^2-x_2^2+x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}-x_2\begin{pmatrix}0&i\\-i&0\end{pmatrix}+x_3\begin{pmatrix}-i&0\\0&i\end{pmatrix}$ | $z_2=x_0-ix_1-jx_2+kx_3$ | {+ – – +} |
| $ds_3^2=x_0^2+x_1^2+x_2^2-x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&i\\i&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ | $z_3=x_0+ix_1+jx_2-kx_3$ | {+ + + –} |
| $ds_4^2=x_0^2+x_1^2-x_2^2-x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ | $z_4=x_0+ix_1-jx_2-kx_3$ | {+ + – –} |
| $ds_5^2=-x_0^2+x_1^2+x_2^2-x_3^2$ | $-x_0\begin{pmatrix}-1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&i\\i&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$ | $z_5=-x_0+ix_1+jx_2-kx_3$ | {– + + –} |
| $ds_6^2=x_0^2-x_1^2-x_2^2-x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}-x_2\begin{pmatrix}0&-i\\i&0\end{pmatrix}-x_3\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ | $z_6=x_0-ix_1-jx_2-kx_3$ | {+ – – –} |
| $ds_7^2=x_0^2+x_1^2-x_2^2+x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ | $z_7=x_0+ix_1-jx_2+kx_3$ | {+ + – +} |
| $ds_8^2=x_0^2-x_1^2+x_2^2+x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ | $z_8=x_0-ix_1+jx_2+kx_3$ | {+ – + +} |
| $ds_9^2=-x_0^2-x_1^2-x_2^2+x_3^2$ | $-x_0\begin{pmatrix}-1&0\\0&1\end{pmatrix}-x_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}-x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $z_9=-x_0-ix_1-jx_2+kx_3$ | {– – – +} |
| $ds_{10}^2=-x_0^2-x_1^2+x_2^2-x_3^2$ | $-x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$ | $z_{10}=-x_0-ix_1+jx_2-kx_3$ | {– – + –} |
| $ds_{11}^2=-x_0^2+x_1^2+x_2^2+x_3^2$ | $-x_0\begin{pmatrix}-1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&i\\i&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $z_{11}=-x_0+ix_1+jx_2+kx_3$ | {– + + +} |
| $ds_{12}^2=x_0^2-x_1^2+x_2^2-x_3^2$ | $x_0\begin{pmatrix}1&0\\0&1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ | $z_{12}=x_0-ix_1+jx_2-kx_3$ | {+ – + –} |
| $ds_{13}^2=-x_0^2-x_1^2+x_2^2+x_3^2$ | $-x_0\begin{pmatrix}-1&0\\0&1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $z_{13}=-x_0-ix_1+jx_2+kx_3$ | {– – + +} |
| $ds_{14}^2=x_0^2-x_1^2+x_2^2+x_3^2$ | $-x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix}+x_1\begin{pmatrix}0&1\\-1&0\end{pmatrix}-x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$ | $z_{14}=-x_0+ix_1+jx_2+kx_3$ | {– + – +} |
| $ds_{15}^2=-x_0^2+x_1^2-x_2^2+x_3^2$ | $-x_0\begin{pmatrix}-1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&-1\\1&0\end{pmatrix}-x_2\begin{pmatrix}0&-1\\-1&0\end{pmatrix}-x_3\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $z_{15}=-x_0+ix_1-jx_2-kx_3$ | {– + – –} |
| $ds_{16}^2=-x_0^2-x_1^2-x_2^2-x_3^2$ | $-x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix}-x_1\begin{pmatrix}0&-1\\-1&0\end{pmatrix}-x_2\begin{pmatrix}0&-i\\i&0\end{pmatrix}-x_3\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$ | $z_{16}=-x_0-ix_1-jx_2-kx_3$ | {– – – –} |

The spin-tensor representations of all 16 quadratic forms given in Table 1 also branch out (degenerate). In a number of cases, the discrete degeneracy (i.e., hidden ambiguity) of the initial ideal state of the $\lambda_{m,n}$-vacuum, when deviating from ideality, can lead to splitting (quantization) into a discrete set of unequal states of its transverse layers.

Sixteen types of $A_4$-matrices are equivalent to 16 "colored" quaternions (section 5.9 in [1]). For clarity, all types of $A_4$-matrices and all varieties of "colored" quaternions are summarized in Table 2.

The Algebra of Signature relates a zero-balanced superposition of linear forms with all 16 possible stignatures:

$$\begin{aligned} ds_\Sigma = {} & (-dx_0-dx_1-dx_2-dx_3) + (dx_0+dx_1+dx_2+dx_3) + \\ & + (dx_0+dx_1+dx_2-dx_3) + (-dx_0-dx_1-dx_2+dx_3) + \\ & + (-dx_0+dx_1+dx_2-dx_3) + (dx_0-dx_1-dx_2+dx_3) + \\ & + (dx_0+dx_1-dx_2+dx_3) + (-dx_0-dx_1+dx_2-dx_3) + \\ & + (-dx_0-dx_1+dx_2+dx_3) + (dx_0+dx_1-dx_2-dx_3) + \\ & + (dx_0-dx_1+dx_2+dx_3) + (-dx_0+dx_1-dx_2-dx_3) + \\ & + (-dx_0+dx_1-dx_2+dx_3) + (dx_0-dx_1+dx_2-dx_3) + \\ & + (dx_0-dx_1-dx_2-dx_3) + (-dx_0+dx_1+dx_2+dx_3) = 0, \end{aligned} \tag{65}$$

with one of the variants of the superposition of sixteen $A_4$-matrices, which also satisfies the vacuum balance condition:

$$\begin{aligned} & x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix}+x_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}+x_2\begin{pmatrix}0&-1\\1&0\end{pmatrix}+x_3\begin{pmatrix}i&0\\0&-i\end{pmatrix} \\ & + x_0\begin{pmatrix}-1&0\\0&1\end{pmatrix}+x_1\begin{pmatrix}0&-1\\1&0\end{pmatrix}+x_2\begin{pmatrix}0&-i\\i&0\end{pmatrix}+x_3\begin{pmatrix}-i&0\\0&i\end{pmatrix} + \\ & + x_0\begin{pmatrix}1&0\\0&-1\end{pmatrix}+x_1\begin{pmatrix}0&i\\-i&0\end{pmatrix}+x_2\begin{pmatrix}0&1\\-1&0\end{pmatrix}+x_3\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \end{aligned} \tag{66}$$

$$+\ x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_3\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+x_1\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}+x_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_3\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}+x_3\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+$$

$$+x_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+x_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_3\begin{pmatrix}-i & 0\\ 0 & -i\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_3\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+x_1\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}+x_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_3\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+x_1\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}+x_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_3\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}+$$

$$+\ x_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+x_1\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}+x_2\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}+x_3\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+$$

$$+x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_3\begin{pmatrix}-i & 0\\ 0 & -i\end{pmatrix}+$$

$$+x_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+x_1\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+$$

$$+x_0\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+x_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+x_2\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}+x_3\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

The stignature-spin-tensor mathematical apparatus presented here is convenient for solving a number of problems related to multilayer inside vacuum rotational processes, which will be considered in the following articles of this proposed project.

**2.11 Using spin-tensors with different signatures**

Let's consider two examples using spin-tensors.

**Example 1:** Let a column matrix and its Hermitian conjugate row matrix be given

$$\begin{pmatrix}s_1\\ s_2\end{pmatrix},\ (s_1^*, s_2^*), \qquad (67)$$

which describe the state of the spinor.

The spin projections on the coordinate axis for the case when the metric 4-space has the signature $(+---)$ can be determined using spin-tensor (67) and $A_4$-matrices (59)

$$(s_1^*, s_2^*)\begin{pmatrix}x_0+x_3 & x_1+ix_2\\ x_1-ix_2 & x_0-x_3\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}=$$
$$=x_0(s_1^*, s_2^*)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}-x_1(s_1^*, s_2^*)\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}-x_2(s_1^*, s_2^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}-x_3(s_1^*, s_2^*)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}= \qquad (68)$$
$$=(s_1^*s_1+s_2^*s_2)x_0-(-s_2^*s_1-s_2^*s_1)x_1-(is_2^*s_1-is_1^*s_2)x_2-(-s_1^*s_1+s_2^*s_2)x_3.$$

**Example 2**: Let the forward and reverse waves be described by expressions

$$\vec{E}_1^{(+)} = \bar{a}_+ e^{-i\frac{2\pi}{\lambda}(ct-r)}, \tag{69}$$

$$\vec{E}_2^{(-)} = \bar{a}_- e^{i\frac{2\pi}{\lambda}(ct-r)}, \tag{70}$$

where $a_+$ and $a_-$ are the amplitudes of the forward and reverse waves. In general, these are complex numbers:

$$\bar{a}_+ = a_+ e^{i\phi_+}, \quad \bar{a}_- = a_- e^{-i\phi_-}, \quad \bar{a}_+^* = a_+ e^{-i\phi_+}, \quad \bar{a}_-^* = a_- e^{i\phi_-}, \tag{71}$$

which contain information about the phases of the waves $\varphi_+$ and $\varphi_-$ .

Mutually opposite waves (69) and (70) can be represented as a two-component spinor:

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = |\psi\rangle = \begin{pmatrix} \bar{a}_+ e^{-i\frac{2\pi}{\lambda}(ct-r)} \\ \bar{a}_- e^{i\frac{2\pi}{\lambda}(ct-r)} \end{pmatrix} \tag{72}$$

and its Hermitian conjugate spinor

$$(s_1^*, s_2^*) = |\psi\rangle^+ = \langle\psi| = \left(\bar{a}_+^* e^{i\frac{2\pi}{\lambda}(ct-r)}, \quad \bar{a}_-^* e^{-i\frac{2\pi}{\lambda}(ct-r)}\right). \tag{73}$$

The normalization condition in this case is expressed by the equality

$$(s_1^*, s_2^*) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \langle\psi|\psi\rangle = \left(\bar{a}_+^* e^{i\frac{2\pi}{\lambda}(ct-r)} \quad \bar{a}_-^* e^{-i\frac{2\pi}{\lambda}(ct-r)}\right) \begin{pmatrix} \bar{a}_+ e^{-i\frac{2\pi}{\lambda}(ct-r)} \\ \bar{a}_- e^{i\frac{2\pi}{\lambda}(ct-r)} \end{pmatrix} = |\bar{a}_+|^2 + |\bar{a}_-|^2. \tag{74}$$

To find the projections of the spin (circular polarization) of a light beam on the coordinate axes, we use the spin-tensor

$$\mathrm{A}_3 = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} = x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{75}$$

which is related to the 3-dimensional metric

$$det(\mathrm{A}_3) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}_{det} = \begin{vmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{vmatrix} = -(x_1^2 + x_1^2 + x_1^2), \tag{76}$$

with signature $(---)$.

Assuming in Ex. (75) $x_1 = x_2 = x_3 = 1$, we consider the spin projections on the coordinate axes

$$(s_1^*, s_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + (s_1^*, s_2^*) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + (s_1^*, s_2^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} =$$

$$= (s_2^* s_1 + s_2^* s_1) + (-is_2^* s_1 + is_1^* s_2) + (s_1^* s_1 - s_2^* s_2). \tag{77}$$

Substituting spinors (72) and (73) into this expression, we obtain the following three spin projections on the corresponding coordinate axes $x_1 = x$, $x_2 = y$, $x_3 = z$:

$$\langle s_x\rangle = \langle\psi|-\sigma_1|\psi\rangle = (s_1^*, s_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} =$$

$$= \left(\bar{a}_+^* e^{i\frac{2\pi}{\lambda}(ct-r)}, \quad \bar{a}_-^* e^{-i\frac{2\pi}{\lambda}(ct-r)}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_+ e^{-i\frac{2\pi}{\lambda}(ct-r)} \\ \bar{a}_- e^{i\frac{2\pi}{\lambda}(ct-r)} \end{pmatrix} = \bar{a}_-^* \bar{a}_+ e^{-i\frac{4\pi}{\lambda}(ct-r)} + \bar{a}_+^* \bar{a}_- e^{i\frac{4\pi}{\lambda}(ct-r)}; \tag{78}$$

$$\langle s_y\rangle = \langle\psi|-\sigma_2|\psi\rangle = (s_1^*, s_2^*) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} =$$

$$= \left(\bar{a}_+^* e^{i\frac{2\pi}{\lambda}(ct-r)}, \quad \bar{a}_-^* e^{-i\frac{2\pi}{\lambda}(ct-r)}\right) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_+ e^{-i\frac{2\pi}{\lambda}(ct-r)} \\ \bar{a}_- e^{i\frac{2\pi}{\lambda}(ct-r)} \end{pmatrix} =$$

$$= \bar{a}_-^* \bar{a}_+ e^{-i\frac{4\pi}{\lambda}(ct-r)} + \bar{a}_+^* \bar{a}_- e^{i\frac{4\pi}{\lambda}(ct-r)} = i\left[\bar{a}_+^* \bar{a}_- e^{i\frac{4\pi}{\lambda}(ct-r)} - \bar{a}_-^* \bar{a}_+ e^{-i\frac{4\pi}{\lambda}(ct-r)}\right]; \tag{79}$$

$\langle s_z \rangle = \langle \psi | -\sigma_3 | \psi \rangle = (s_1^*, s_2^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} =$

$$= \left( \bar{a}_+^* e^{i\frac{2\pi}{\lambda}(ct-r)}, \quad \bar{a}_-^* e^{-i\frac{2\pi}{\lambda}(ct-r)} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{a}_+ e^{-i\frac{2\pi}{\lambda}(ct-r)} \\ \bar{a}_- e^{i\frac{2\pi}{\lambda}(ct-r)} \end{pmatrix} = |\bar{a}_+|^2 - |\bar{a}_-|^2. \quad (80)$$

In the case of $\varphi_+ = \varphi_- = 0$, Formulas (78) – (80) take the following simplified form:

$$\langle s_x \rangle = 2a_+ a_- \cos\left[\frac{4\pi}{\lambda}(ct - r)\right] = 2a_+ a_- \cos[2(\omega t - kr)], \quad (81)$$

$$\langle s_y \rangle = 2a_+ a_- \sin\left[\frac{4\pi}{\lambda}(ct - r)\right] = 2a_+ a_- \sin[2(\omega t - kr)],$$

$$\langle s_z \rangle = |a_+|^2 - |a_-|^2.$$

In the case of equality of the amplitudes of the direct and backward waves $a_+ = a_-$, instead of Eqs. (81), we obtain the following average spin projections

$$\langle s_x \rangle = 2a_+^2 \cos[2(\omega t - kr)], \quad (82)$$

$$\langle s_y \rangle = 2a_+^2 \sin[2(\omega t - kr)],$$

$$\langle s_z \rangle = 0.$$

The projection of the spin (the rotating vector of the electric field strength) on the direction of propagation of the light beam Z is unchanged and equal to zero. At the same time, its projection onto the XY plane, perpendicular to the direction of propagation of this beam, rotates around the Z axis with an angular velocity $\omega = 4\pi c/\lambda$. Thus, the spinor representation of the propagation of a conjugated pair of waves leads to a description of circular polarization without resorting to additional hypotheses.

Similarly, can be performed an analysis of wave propagation in a 3-dimensional metric extent with signatures:

(– – –), (+ – –), (– + –), (– – +), (+ + +), (– + +), (+ – +), (+ + –).

### 2.12 The Dirac bundle of quadratic form

Let's consider the Dirac "bundle" of a quadratic form using the example of the metric

$$ds^2 = c^2dt^2 + dx^2 + dy^2 + dz^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \text{ with signature } (+ + + +). \quad (83)$$

We imagine this metric as a product of two affine (linear) forms

$$ds^2 = ds'ds'' = (\gamma_0 dx_0' + \gamma_1 dx_1' + \gamma_2 dx_2' + \gamma_3 dx_3') \cdot (\gamma_0 dx_0'' + \gamma_1 dx_1'' + \gamma_2 dx_2'' + \gamma_3 dx_3''). \quad (84)$$

By opening the brackets in this expression, we get

$$ds'ds'' = \sum_{\mu=0}^{3} \sum_{\eta=0}^{3} \gamma_\mu \gamma_\eta dx^\mu dx^\eta = \frac{1}{2} \sum_{\mu=0}^{3} \sum_{\eta=0}^{3} (\gamma_\mu \gamma_\eta + \gamma_\eta \gamma_\mu) dx^\mu dx^\eta. \quad (85)$$

There are at least two options for determining the values $\gamma_\mu$ that satisfy the condition of equality of Exs. (83) – (85): 1) the method of Clifford aggregates (for example, quaternions); 2) the Dirac method.

In the case of applying the Clifford aggregates method, the linear forms included in expression (84) are represented as a pair of affine aggregates:

$$ds' = \gamma_0 cdt' + \gamma_1 dx' + \gamma_2 dy' + \gamma_3 dz', \quad (86)$$
$$ds'' = \gamma_0 cdt'' + \gamma_1 dx'' + \gamma_2 dy'' + \gamma_3 dz'', \quad (87)$$

with stignature {+ + + +}, where $\gamma_\mu$ are objects that satisfy the commutative condition of the Clifford algebra

$$\gamma_\mu\gamma_\eta + \gamma_\eta\gamma_\mu = 2\delta_{\mu\eta}, \qquad (88)$$

where $\delta_{\mu\eta} = \begin{cases} 1 \text{ for } \mu = \eta, \\ 0 \text{ for } \mu \neq \eta \end{cases}$ are the Kronecker symbols. (89)

In the second case, the Dirac method suggests using the identity matrix instead of the Kronecker symbols (89)

$$\delta_{\mu\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (90)$$

then condition (88) is satisfied, for example, by the following set of 4×4 Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \qquad (91)$$

Эти матрицы можно рассматривать в качестве образующих соответствующей алгебры Клиффорда. В этом случае выражение (85) приобретает матричный вид

$$(ds_{ii}^2) = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3}\gamma_\mu\gamma_\eta dx^\mu dx^\eta = \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\eta=0}^{3}(\gamma_\mu\gamma_\eta + \gamma_\eta\gamma_\mu)dx^\mu dx^\eta, \qquad (92)$$

where $$(ds_{ii}^2) = \begin{pmatrix} ds_{00}^2 & 0 & 0 & 0 \\ 0 & ds_{11}^2 & 0 & 0 \\ 0 & 0 & ds_{22}^2 & 0 \\ 0 & 0 & 0 & ds_{33}^2 \end{pmatrix}. \qquad (93)$$

Ex. (92), taking into account (90), can be represented as

$$(ds_{ii}^2) = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3}\gamma_\mu\gamma_\eta dx^\mu dx^\eta = c^2dt^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + dx^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} +$$

$$+dy^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + dz^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (94)$$

Let’s return to the quadratic form (83) and its Dirac bundle (92)

$$(ds_{ii}^2) = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3}\gamma_\mu\gamma_\eta dx^\mu dx^\eta = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3} b_{\mu\eta} dx^\mu dx^\eta, \qquad (95)$$

where $$\gamma_\mu\gamma_\eta = b_{\mu\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (96)$$

We consider all possible ways of writing Ex. (95). To do this, we use the following basis of 16 possible Dirac $\gamma_\mu^{(\rho)}$-matrices:

$$\gamma_0{}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_1{}^{(0)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \gamma_2{}^{(0)} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \gamma_3{}^{(0)} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \quad (97)$$

$$\gamma_0{}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \gamma_1{}^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \gamma_2{}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \gamma_3{}^{(1)} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_0{}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \gamma_1{}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \gamma_2{}^{(2)} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \gamma_3{}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_0{}^{(3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_1{}^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \gamma_2{}^{(3)} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \gamma_3{}^{(3)} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Dirac's method, in contrast to the method of affine aggregates, allows one to simultaneously "stratify" four metric spaces with four metrics that are components of the matrix (93).

In the Algebra of Signatures, sixteen quadratic forms (31) with corresponding signatures (32) are considered, each of them can also be "stratify" by the Dirac method

$$\left(ds_{ii}^{(a,b)2}\right) = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3}\gamma_\mu^{(a)}\gamma_\eta^{(b)}dx^\mu dx^\eta, \quad (98)$$

here $\gamma_\mu{}^{(a)}\gamma_\eta{}^{(b)} = b_{\mu\eta}{}^{(ab}$, (99)

But in this case, each $b_{\mu\eta}{}^{(ab)}$-matrix has a corresponding stignature: (100)

$$b_{\mu\eta}^{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b_{\mu\eta}^{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{20} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b_{\mu\eta}^{01} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b_{\mu\eta}^{11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b_{\mu\eta}^{21} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b_{\mu\eta}^{31} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b_{\mu\eta}^{02} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b_{\mu\eta}^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b_{\mu\eta}^{03} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{13} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{23} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad b_{\mu\eta}^{33} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

T

he signs before the units in the diagonal $b_{\mu\eta}{}^{(ab)}$-matrices correspond to the sets of signs in the components of the signature matrix (32). In this paragraph, for brevity, we will temporarily omit the upper indices and instead of "$b_{\mu\eta}{}^{(ab)}$-matrix" we will write "$b_{\mu\eta}$-matrix".

Let's return to the Dirac "bundle" of the quadratic form (92)

$$\left(ds_{ii}^2\right) = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3}\gamma_\mu\gamma_\eta dx^\mu dx^\eta = \sum_{\mu=0}^{3}\sum_{\eta=0}^{3} b_{\mu\eta} dx^\mu dx^\eta, \quad (101)$$

where

$$\gamma_\mu\gamma_\eta = b_{\mu\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (102)$$

Let's return to the Dirac "bundle" of the quadratic form (92) и рассмотрим всевозможные варианты ее раскрытия.

Each of the sixteen $\gamma_{\mu}^{(\rho)}$-matrices (97) can be selected a second $\gamma_{\chi}^{(\tau)}$-matrix from the same set, such that their product is equal to the $b_{\mu\eta}$-matrix (102). For example:

$$\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{103}$$

Each $\gamma_{\mu}^{(\rho)}$-matrix (97) can have one of 16 possible stignatures. For example:

$$\gamma_{11}^{00} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{30} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{104}$$

$$\gamma_{11}^{01} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{11} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{21} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{31} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{11}^{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{12} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{22} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{32} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{11}^{03} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{23} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_{11}^{33} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For each of these $\gamma_{\mu\rho}^{ij}$-matrices, it is also possible to select a second $\gamma_{\chi\tau}^{nj}$-matrix, the product of which leads to the $b_{\mu\eta}$-matrix (102).

Thus, taking into account 16 stignatures from 16 $\gamma_{\mu\rho}$-matrices (97), 16×16 = 256 $\gamma_{\mu\rho}^{ij}$-matrices are obtained. Each $\gamma_{\mu\rho}^{ij}$-matrix (104) can be transformed into one of 16 mixed matrices. Let us explain this statement by the example of the $\gamma_{11}^{13}$-matrix:

$${}_{00}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{10}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{20}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{30}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$${}_{01}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{11}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{21}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{31}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$${}_{02}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad {}_{12}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad {}_{22}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad {}_{32}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$${}_{03}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad {}_{13}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad {}_{23}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad {}_{33}\gamma_{11}^{13} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{105}$$

With a similar stirring of all 256 $\gamma_{\mu\rho}^{ij}$-matrices (105), a basis of 163 = 256 × 16 = 4096 ${}_{nk}\gamma_{\mu\rho}^{ij}$-matrices is obtained. Therefore, in this case is the $b_{\mu\eta}$-matrix (102) can be given by one of 4096 products of pairs of ${}_{nk}\gamma_{\mu\rho}^{ij}$--matrices.

In turn, all sixteen $b_{\mu\eta}$-matrices (100) can be given by $16^4$ = 65536 different variants of paired products of ${}^{vc}{}_{nk}\gamma_{lm}{}^{ij}$-matrices. Similarly, it is possible to continue building up the basis of generalized Dirac $\gamma$-matrices almost indefinitely.

The Dirac "bundle" of only one quadratic form (83) was considered above. Similarly, all other metrics (31) are "stratified".

The whole set of ${}^{vc}{}_{nk}\gamma_{lm}{}^{ij}$-matrices will be called generalized Dirac matrices, and the metric stratified by means of these matrices will be called a Dirac bundle of quadratic form with the corresponding signature.

**3 Conclusions**

In this second part of "Geometrized Vacuum Physics" there are no physical models. This article is devoted to the development of the mathematical apparatus of the Algebra of Signatures, which follows from the Algebra of Stignatures [1].

The Algebra of Stignatures and the Algebra of Signatures are a kind of mental glasses that it is suggested to put on the researcher's mind in order to recognize the Meanings realized in the reality around us.

For some researchers, it will be important to know that the Algebra of Stignatures and the Algebra of Signatures (under the common name Algebra of Signatures, or abbreviated "Alsigna") is an extension of the ancient Pythagorean tradition (i.e., scientific knowledge) based on the Algorithms for revealing the Great Name of the ALMIGHTY י-ה-ו-ה (Yud-Key-Vav-Key) [6], underlying Judaism, and supplemented by the logical constructions of Taoism, Hinduism, Zoroastrianism and Ometeotl.

Algebra of Signatures is open for its replenishment and expansion based on the logical concepts of various religions, cultures and philosophical schools. The mathematical apparatus of the Algebra of Signatures can be developed by representatives of all ancient philosophical traditions, with the urgent observance of the condition of "vacuum (i.e. zero) balance". In this sense, Algebra of Signatures can serve as a universal scientific platform for general cognitive "Agreement".

In this article, pairwise scalar multiplication of vectors from all 16 affine spaces with 4-bases shown in Figure 3, led to the formation of 16 × 16 = 256 metric 4-spaces with 4-metrics of the form (10), which intersect at the point $O$ under study (see Figure 1).

Among 256 metric spaces, there were 16 types of spaces with corresponding signatures, forming a matrix of signatures (32)

$$stign\left(ds^{(a,b)2}\right) = \begin{pmatrix} (+++\,+) & (+++\,-) & (-++\,-) & (++-\,+) \\ (---\,+) & (-++\,+) & (--+\,+) & (-+-\,+) \\ (+--\,+) & (++-\,-) & (+--\,-) & (+-+\,+) \\ (--+\,-) & (+-+\,-) & (-+-\,-) & (---\,-) \end{pmatrix}.$$

The properties of this matrix of signatures largely repeat the properties of the matrix of stignatures obtained in the article [1].

Further, it was shown that the signature of a metric space is related to its topology, and the additive imposition of 256 metric spaces with 16 types of topologies (or signatures) satisfies the vacuum balance condition.

At the same time, it turned out that the mathematical apparatus of the Algebra of Signatures allows the additive imposition of an infinite number of metric spaces with 16 types of topologies under the condition of a vacuum (i.e., zero) balance, which leads to the formation of a Ricci flat space similar to a Calabi-Yau manifold.

At the end of the article, a spin-tensor representation of metrics with different signatures is considered and a Dirac bundle of quadratic forms is presented to describe complex rotational intra-vacuum processes.

Alsigna's mathematical apparatus developed here and in the previous article [1] will be used in subsequent articles of this project to describe and mathematically model many vacuum effects and other physical phenomena.

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