On analogy of black hole and phase singularities

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We formulate the eikonal equation in (3+1)-dimensional spherically symmetric curved space-time using Clebsch variables. Black hole singularity is considered as the extremely high speed changing of the eikonal. We investigate the analogy of a black hole singularity and the phase singularity in optics known as optical vortex.

Keywords: *eikonal equation, null geodesic, Schwarzschild metric, black hole singularity, phase singularity, vortex.*

It might be there exists a relation between the phenomenon of an extremely high refractive index when the refractive index tends towards infinity for light ray propagation at a distance of a black hole event horizon or Schwarzschild radius (a black hole singularity), and the phenomenon of the zero-intensity (phase singularity) in optics. Zeros of field intensity typically manifest as lines in three-dimensional space, around which the phase has circulating or helical behaviour known as optical vortices[1].

The static spherically symmetric curved space-time (gravitational field) described by the Schwarzschild metric[2], an interval, can be written as

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2+\left(1-\frac{r_s}{r}\right)^{-1}dr^2+r^2\ d\theta^2 + r^2\sin^2\theta\ d\phi \tag{1}$$

where r is the spatial (radial) coordinate, the distance from the centre of a mass of the massive body (such as a black hole, a massive star) to the points (these points have the same distance from the centre) where the Schwarzschid metric is being evaluated, and

$$r_s = \frac{2GM}{c^2} \tag{2}$$

is the Schwarzschild radius, M is the mass of the spherically symmetric body, G is the gravitational constant, and c is the speed of light in flat space-time. Equation (1) is also known as the Schwarzschild solution[3] (to Einstein field equation). The Schwarzschild solution holds outside the surface of the massive body that is producing the gravitational field, where there is no matter[3].

Let us write the eikonal equation as[4]

$$|\partial_\mu\psi(\vec{r})| = n(\vec{r}) \tag{3}$$

where $\psi(\vec{r})$ is the eikonal, $n(\vec{r})$ is the refractive index. In our previous work[4], by applying null geodesic $ds^2 = 0$ to the Schwarzschild metric (1), we found that the eikonal equation in (3+1)-dimensional spherically symmetric curved space-time can be written below

$$\left|\partial_\mu\left\{\frac{c}{f_\theta}\int f\ \partial_\nu q\ f^*\ dx^\nu + ct\right\}\right| = \left(1-\frac{r_s}{r}\right)^{-1} \tag{4}$$

where $f\ \partial_\nu q = \vec{A}_\nu$ is the $U(1)$ gauge potential, f is a function of amplitude, q is the phase, both are Clebsch variables (scalars), f^* is the conjugate complex of f, and f_θ is angular frequency. Eq.(4) shows explicitly that the length (a space interval), the eikonal, can be related to the $U(1)$ gauge potential which represents the existence of the light ray. It also shows that the length is curved due to the mass of the spherically symmetric body that is producing the spherically symmetric gravitational field.

If we take $r = r_s$ then eq.(4) becomes

$$\left|\partial_\mu\left\{\frac{c}{f_\theta}\int f\ \partial_\nu q\ f^*\ dx^\nu + ct\right\}\right| = \infty \tag{5}$$

Eq.(5) shows us that there exists the singularity. The phase, $q(\vec{r},t)$, is nothing but an angle. The dimension of terms in the curly bracket of eq.(5) is a length, a scalar. If the gradient applies to such scalar then we obtain a vector. The magnitude of this vector is related to the infinity. It means that the infinity (the singularity) is related to the extremely high speed changing of the eikonal length. It has a consequence that the value of the eikonal (phase) is not unique, it has no well-defined value. The eikonal (phase) is indeterminate.

In optics, there exists the phase singularity phenomenon, i.e. the phase $q(\vec{r},t)$ is indeterminate[5]. The nature of the singularity is determined by the fact that the wave field[6], ϕ, is a smooth single-valued function of its variables[5]. The wave field can be written as[5,6]

$$\phi(\vec{r},t) = \rho(\vec{r},t)\ e^{iq(\vec{r},t)} \tag{6}$$

where $\rho(\vec{r},t)$ is the amplitude and the phase can be written as[4,6]

$$q(\vec{r},t) = X\left\{\psi(\vec{r}) - ct\right\} \tag{7}$$

where $X = f_\theta/c$, and $\psi(\vec{r})$ is the eikonal as the function of coordinates only. It means that eikonal, $\psi(\vec{r})$, is the special case of the phase, $q(\vec{r},t)$, when there is no time dependence.

Eq.(6) shows that the indeterminate value of the phase has the consequence that the smooth single-valued properties of the wave field, $\phi(\vec{r},t)$, depends only on the amplitude properties. The amplitude should be a smooth single-valued function. We choose the value of the amplitude is zero to satisfy the smooth and single-valued properties of the wave field. Non-zero choice of the amplitude will break the single-valued property of the wave field. Zero amplitude is related to zero intensity[1,5].

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