Two totally connected superluminal Natario warp drive spacetimes with variable velocities

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Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However one the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1 + 1$ spacetime with variable velocities and with or without lapse functions using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline the final results found in our work which are the following ones: For both cases with variable velocities and with or without the lapse function the Horizon do not exists at all.Due to the extra terms in the lapse function and in the variable velocities that affects the whole spacetime geometry these solutions allows to circumvent the problem of the Horizon.

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1 Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8],[9],[10] and mainly [11] and [23]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7],[8] and mainly [11] and [21]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7],[8] and mainly [20] and [24]).In [20] and [24] we present totally connected Natario warp drive spacetimes.See the results presented in section 5 in [20] and section 5 in [24].

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8],[9],[10],[11],[20],[21],[23] and [24]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

[1] do not violates Relativity

Alcubierre([16]) used the so-called 3+1 Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([15]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [16], see also eq 1 pg 3 in [1])[2][3] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.

However some important things must be outlined in both the Alcubierre or Natario warp drive spacetimes:

- 1)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs.They do not accelerate or de-accelerate and travel always with a constant speed.But a real warp drive must "know" how to accelerate for example from 0 to a speed of 200 times faster than light in the beginning of an interstellar journey and in the end of the journey it must de-accelerate again to 0 in the arrival at the destination point which means to say of course a distant star.

- 2)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs raised to the square in their equations for the negative energy density.An accelerating warp drive probably must have the terms of variable velocities or accelerations included in the expression for the negative energy density since this energy is responsible for the generation of the warp drive spacetime.

- 3)-The warp drives as proposed by Alcubierre or Natario always have the so-called lapse function of the ADM formalism always equal to 1

Since the Natario vector is the generator of the Natario warp drive spacetime metric in this work we present the original Natario vector but including the coordinate time as a new Canonical Basis for the Hodge star generating an expanded Natario vector and an extended Natario warp drive spacetime metric which encompasses accelerations and variable velocities.Our proposed extended Natario warp drive metric with variable velocity vs due to a constant acceleration a is given by the following equation:(see Appendix F in [12])

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta) dt^2 + 2(X_{rs} drs + X_\theta d\theta) dt - drs^2 - rs^2 d\theta^2 \tag{1}$$

Note that in this equation a new set of contravariant and covariant components X^t and X_t appears because in this case as the velocity vs changes its value as times goes by due to a constant acceleration a this affects the whole spacetime geometry.

Two important things must be outlined by now:

- 1)-The Natario shape function used in the equation with constant speed(see Appendix E in [12]) is valid also in the equation with variable speed.

- 2)-This equations also satisfies the Natario criteria for a warp drive spacetime.

[2] see also Appendix E in [12]

[3] see the Remarks section on our system to quote pages in bibliographic references

In this work we present the new extended equations for the Natario warp drive spacetime which encompasses accelerations and variable speeds with or without lapse function using also the ADM formalism and we arrive at the conclusion that the new equations are also valid solutions for the warp drive spacetime according to the Natario criteria.

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pgs $[106(a)], [98(b)]$ in [17] and pgs $[394(a)][377(b)]$ in [18]).

What Alcubierre and Natario did was an attempt to reproduce the expansion of the Universe in a local way creating a local spacetime distortion that expands the spacetime behind a ship and contracts spacetime in front reproducing the superluminal expansion of the Universe moving away the departure point in an expansion and bringing together the destination point in a contraction.The expansion-contraction can be seen in the abs of the original Alcubierre paper in [1].Although Natario says in the abs of his paper in [2] that the expansion-contraction does not occurs in its spacetime in pg 5 of the Natario paper we can see the expansion-contraction occurring however the expansion of the normal volume elements or the trace of the extrinsic curvature is zero because the contraction in the radial direction is exactly balanced by the expansion in the perpendicular directions.

An excellent explanation on how a spacetime distortion or a perturbation pushes away a spaceship from the departure point and brings the ship to the destination point at faster than light speed can be seen at pg 34 in [3],pgs $[260(a)260(b)][261(a)261(b)]$ in [4].Note that in these works it can be seen that the perturbation do not obeys the time dilatation of the Lorentz transformations hence the speed limit of Special Relativity cannot be applied here.

An accelerated warp drive can only exists if the astronaut in the center of the warp bubble can somehow communicate with the warp bubble walls sending instructions to change its speed.But for signals at light speed the Horizon exists at least for the warp drive with constant velocity.So light speed cannot be used to send signals to the front of the bubble.(see pg 16 in [7] and pg 21 in [8]).Besides in the Natario warp drive with constant velocity the negative energy density covers the entire bubble.(see Appendices B,C and D in [23]).Since the negative energy density have repulsive gravitational behavior the photon of light if possible to reach the bubble walls would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls(see pg $[116(a)][116(b)]$ in [13])

The solution that allows contact with the bubble walls was presented in pg 28 in [7] and pg 31 in [8]. Although the light cone of the external part of the warp bubble is causally disconnected from the astronaut who lies inside the large bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro bubble so these bubbles can be "engineered" to be sent to the large bubble. This idea seems to be endorsed by pg 34 in [3],pgs $[268(a)268(b)]$ in [4] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.See also the results presented in section 5 in [20] and section 5 in [24] for a totally connected superluminal Natario warp drive spacetime.

Horizons were deeply covered in the warp drive literature but always for constant velocities.(see pg 6 in [2],pg 16 in [7],pg 21 in [8],pg 34 in [3],pgs $[268(a)268(b)]$ in [4])(see also section 4 in [20] and section 4 in [24]).The behavior of a photon sent to the front of the warp bubble in the case of a warp drive always with variable velocity and with or without lapse function is the main purpose of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ spacetime with both variable velocities and with or without lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline(or underline or reinforce) the final results found in our work which are the following ones:

- 1)-In the case of the Natario warp drive with variable velocity and without the lapse function the Horizon do not exists exists as expected and in agreement with the current literature(see section 5 in [20]).

- 2)-In the case of the Natario warp drive with variable velocity and with lapse function the Horizon do not exists at all.Due to the extra term provided by the lapse function that affects the whole spacetime geometry this solution with variable velocities have different results when compared to the variable velocity solution without lapse function.Remember that we are presenting our results using step by step mathematics in order to better illustrate our point of view.

- 3)-Both solutions with variable velocities and with our without lapse function keeps the Natario warp drive totally connected.

A lapse function with values different than 1 adapted to the Natario warp drive that obeys the $3+1$ ADM formalism with variable velocities must possess the following properties:

- inside the warp bubble(flat spacetime where the spaceship is located)the lapse function is equal to 1

- outside the warp bubble(flat spacetime where an external observer watches the ship passing by)the lapse function is also equal to 1

- in the Natario warped region(warp bubble walls curved spacetime) the lapse function must possesses a large value at least greater than or equal to the modulus of the ship velocity to keep the warp bubble totally connected.(see section 5 in [24])

The Natario warp drive equation that obeys the $3+1$ ADM formalism with variable velocities and a lapse function α or N is given below:(see Appendix A in this work)(see Appendix F in [12] with an adaption from Appendix J in [23])(This was also adapted from section 3 in [24])

$$ds^2 = (\alpha^2 - 2\alpha X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \qquad (2)$$

Compare the equation given above with the first equation presented in this work:In the place of 1 we have α as the lapse function.

Inside and outside the bubble (flat spacetime) both equations are mathematically equivalent.The difference occurs in the Natario warped region where the presence of the lapse function affects the whole spacetime geometry.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes.

This work must be regarded as a companion work to our works in [12],[20],[22],[23] and [24] which are required readings in order to understand some of the mathematics used in this text.

In order to avoid confusion between the constant acceleration a used in this work and in refs [12],[20] and [22] and the lapse function α used in refs [23] and [24] we represent the lapse function by the letter N as in Misner-Thorne-Wheeler(MTW)([15]).The Natario warp drive equation that obeys the $3+1$ ADM formalism with variable velocities and a lapse function N is given by:(see Appendix A in this work)

$$ds^2 = (N^2 - 2NX_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{3}$$

2 The equation of the Natario warp drive spacetime metric with a variable speed vs due to a constant acceleration a in the original $3+1$ ADM formalism without a lapse function or with a lapse function N always equal to 1

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix F in [12] for details)

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{4}$$

The equation of the Natario vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{5}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C in [12]):

$$X^t = 2n(rs)rs cos\theta a \tag{6}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{7}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{8}$$

The covariant shift vector components X_t,X_{rs} and X_θ are given by:

$$X_t = 2n(rs)rs cos\theta a \tag{9}$$

$$X_{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{10}$$

$$X_\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2 \sin\theta \tag{11}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx + xdvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs})dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{12}$$

But since $X_t = X^t$ and $X_{rs} = X^{rs}$ the equation can be written as given below:

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs} drs)dt - drs^2 \tag{13}$$

$$X^t = 2n(rs)rsa \tag{14}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{15}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{16}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero[4] and the covariant shift vector $X_{rs} = 2[2n(rs)^2]at$ with $X_{rs} = 0$ inside the bubble and $X_{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

[4]except in the neighborhoods of the bubble radius.See Section 2 in [12]

3 Horizons(causally disconnected portions of spacetime geometry in the equation of the Natario warp drive spacetime metric with a variable speed vs and a constant acceleration a in the original $1+1$ ADM formalism) without the lapse function or with a lapse function always equal to 1

Like the section 4 in [20] the mathematical discussions of this section also uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.(Unlike the section 4 in [20] where from the geometrical point of view the photon stopped in the Horizon and the outermost layers of the bubble were causally disconnected from the observer in the center of the bubble for the case of constant velocity vs) in this section and still from a geometrical point of view we will demonstrate that in the case of variable velocities the photon do not stops in the Horizon and the Horizon do not exists and the outermost layers of the bubble are causally connected to the observer in the center of the bubble.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime:

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{17}$$

The contravariant shift vector components X^t and X^{rs} are then:

$$X^t = 2n(rs)rsa \tag{18}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{19}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{20}$$

The term $1 - 2X^t + (X^t)^2$ in the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime can be simplified as:

$$1 - 2X^t + (X^t)^2 = (1 - (X^t))^2 \tag{21}$$

Hence the equation becomes:

$$ds^2 = ((1 - (X^t))^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{22}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime we have:

$$0 = ((1 - (X^t))^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{23}$$

Dividing both sides by dt^2 we have:

$$0 = ((1 - (X^t))^2 - (X^{rs})^2) + 2(X^{rs}\frac{drs}{dt}) - (\frac{drs}{dt})^2 \tag{24}$$

Making the following algebraic substitution:

$$U = \frac{drs}{dt} \tag{25}$$

We have:

$$0 = ((1 - (X^t))^2 - (X^{rs})^2) + 2(X^{rs}U) - (U)^2 \tag{26}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$(U)^2 - 2(X^{rs}U) - ((1 - (X^t))^2 - (X^{rs})^2) = 0 \tag{27}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X^{rs}) \pm \sqrt{4(X^{rs})^2 + 4((1 - (X^t))^2 - (X^{rs})^2)}}{2} \tag{28}$$

$$U = \frac{2(X^{rs}) \pm \sqrt{4(X^{rs})^2 + 4(1 - (X^t))^2 - 4(X^{rs})^2}}{2} \tag{29}$$

The simplified algebraic expression becomes:

$$U = \frac{2(X^{rs}) \pm \sqrt{4(1 - (X^t))^2}}{2} \tag{30}$$

Which leads to:

$$U = \frac{2(X^{rs}) \pm 2(1 - (X^t))}{2} \tag{31}$$

And the final result is then given by:

$$U = X^{rs} \pm (1 - (X^t)) \tag{32}$$

The above equation have two possible solutions U respectively $U = X^{rs} + (1 - (X^t))$ and $U = X^{rs} - (1 - (X^t))$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X^{rs} - (1 - (X^t)) = X^{rs} + X^t - 1 \tag{33}$$

$$U_{rear} = X^{rs} + (1 - (X^t)) = X^{rs} - X^t + 1 \tag{34}$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X^{rs} - (1 - (X^t)) = X^{rs} + X^t - 1 \tag{35}$$

$$X^t = 2n(rs)rsa \tag{36}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at = 4n(rs)^2 at + 2rsn'(rs)at = 2n(rs)2n(rs)at + 2rsn'(rs)at \tag{37}$$

Note that unlike the section 4 in [20] when we got only one contravariant shift vector component X^{rs} and it was this component that dictated the Horizon behavior of the front solution for the quadratic form now we get two contravariant shift vector components X^{rs} and X^t for the front solution of the quadratic form.This would be algebraically more complicated to be manipulated but fortunately we can write the component X^{rs} in function of the component X^t simplifying greatly the analysis.Using the following algebraic expressions both written in function of X^t

$$\frac{X^t}{rs} = 2n(rs)a \tag{38}$$

$$\frac{X^t}{n(rs)} = 2rsa \tag{39}$$

We can write X^{rs} in function of X^t as follows:

$$X^{rs} = 2n(rs)2n(rs)at + 2rsn'(rs)at = 2n(rs)\frac{X^t}{rs}t + n'(rs)\frac{X^t}{n(rs)}t \tag{40}$$

$$X^{rs} = 2\frac{n(rs)}{rs}tX^t + \frac{n'(rs)}{n(rs)}tX^t \tag{41}$$

Simplifying we get:

$$X^{rs} = tX^t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] \tag{42}$$

And the final solution of the quadratic form for the photon sent to the front part of the bubble is finally given by:

$$U_{front} = X^{rs} + X^t - 1 = tX^t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + X^t - 1 \tag{43}$$

The expression simplified leads to:

$$U_{front} = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 \tag{44}$$

$$U_{front} = (2n(rs)rsa)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 \tag{45}$$

The final solution of the quadratic form for the photon sent to the front part of the bubble is:

$$U_{front} = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 \tag{46}$$

$$U_{front} = (2n(rs)rsa)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 \tag{47}$$

Again note that unlike the section 4 in [20] where the dominant term was the component X^{rs} now the dominant term is the component X^t and $X^t = 2n(rs)rsa$.

Considering again a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) and assuming a continuous behavior for $n(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2n(rs)rsa$ from 0 to rsa we can clearly see that inside the bubble $2n(rs)rsa = 0$ because $n(rs) = 0$ and outside the bubble $2n(rs)rsa = rsa$ because $n(rs) = \frac{1}{2}$ and assuming also continuous values from 0 to rsa then somewhere in the Natario warped region where $0 < n(rs) < \frac{1}{2}$ we have the situation in which $2n(rs)rsa = 1$ because 1 lies in the continuous interval from 0 to rsa and in consequence $X^t = 1$.

The final solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ is:

$$U_{front}(X^t = 1) = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 = t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1 - 1 \tag{48}$$

Simplifying the result leads ourselves to:

$$U_{front}(X^t = 1) = t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] \neq 0!!! \tag{49}$$

Note that unlike the section 4 in [20] the result is not zero !!!.The photon do not stops in the Natario warped region and the Horizon no longer exists!!!.

The place where $X^t = 2n(rs)rsa = 1$ is the place where the Natario shape function is $n(rs) = \frac{1}{2rsa}$ well inside the Natario warped region in which $0 < \frac{1}{2rsa} < \frac{1}{2}$ with $a >= 1$ and of course $rs > 0$.

Rewriting the solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ using the value of the Natario shape function $n(rs) = \frac{1}{2rsa}$ we get:

$$U_{front}(X^t = 1) = t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] = t[2\frac{\frac{1}{2rsa}}{rs} + \frac{n'(rs)}{\frac{1}{2rsa}}] \neq 0!!! \tag{50}$$

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2 a} + 2n'(rs)rsa] \neq 0!!! \tag{51}$$

The solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ using the value of the Natario shape function $n(rs) = \frac{1}{2rsa}$ is:

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2a} + 2n'(rs)rsa] \neq 0!!! \tag{52}$$

Note that both the expressions $\frac{1}{rs^2a}$ and $2n'(rs)rsa$ have fractionary values close to zero but always greater than zero and above everything else not zero at all!!!.In the first expression we have both $rs > 0$ and $a \geq 1$ and in the second expression the derivative of the shape function must have low values in order to reduce the needs of negative energy density to sustain a warp bubble.

Then we can easily see that $0 < \frac{1}{rs^2a} < 1$ due to $rs > 0$ and $a \geq 1$ in the fraction and $0 < 2n'(rs)rsa < 1$ see section 3 in [11] for the low values of the square derivative of the Natario shape function able to reduce the negative energy density requirements implying in a low value for the derivative of the shape function.The expressions can be written as follows:

$$\frac{1}{rs^2a} \neq 0!!! \rightarrow \frac{1}{rs^2a} \simeq 0!!! \rightarrow \frac{1}{rs^2a} > 0!!! \rightarrow rs > 0 \rightarrow a \geq 1 \tag{53}$$

$$2n'(rs)rsa \neq 0!!! \rightarrow 2n'(rs)rsa \simeq 0!!! \rightarrow 2n'(rs)rsa > 0!!! \tag{54}$$

Then the solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ using the value of the Natario shape function $n(rs) = \frac{1}{2rsa}$ is better written as:

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2a} + 2n'(rs)rsa] \cong 0!!! \tag{55}$$

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2a} + 2n'(rs)rsa] > 0!!! \tag{56}$$

The result is close to zero but it is always greater than zero!!!.The photon do not stops but moves at a very low speed!!!.The Horizon do not exists!!!,

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime.We know from the section 4 in [20] that in the case of the Natario warp drive with constant speed the negative energy density covers the entire bubble.(see Appendices B,C and D in [23]).Unfortunately we dont have the distribution of the negative energy density for the case of variable speeds.See section 5 pg 15 in [12] for the considerations of the negative energy density in the solution for variable speeds.Then we dont know if the negative energy density covers the entire bubble in the case of variable speeds but if this happens and since the negative energy density have repulsive gravitational behavior(see pg $[116(a)][116(b)]$ in [13]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which would perhaps exists in the front of the bubble never reaching the bubble walls.

The solution to control the warp bubble would then be similar to the solution presented in the end of the section 4 in [20] using "pre-programmed" micro warp bubbles as described by pg 28 in [7] and pg 31 in [8] resembling the idea outlined in fig 7 pg $[96(a)][83(b)]$ in [14].

Back again to the solution of the photon sent to the front of the bubble:

$$U_{front} = X^{rs} - (1 - (X^{t})) = X^{rs} + X^{t} - 1 \tag{57}$$

This is the solution for a Natario warp drive metric with variable velocities.Compare with the solution of the photon sent to the front of the bubble in a Natario warp drive metric with fixed velocity given in section 4 in [20]:

$$U_{front} = X^{rs} - 1 \tag{58}$$

The term in X^{t} affects the whole structure of the spacetime geometry eliminating once for all the problem of the Horizon. When the velocity is constant the term in X^{t} vanishes leaving only the term is term in X^{rs} and in consequence $X^{rs} - 1$ and as already seen in section 4 in [20] the Horizon appears.

4 The equation of the Natario warp drive spacetime metric with a variable speed vs due to a constant acceleration a in the original $3+1$ ADM formalism with a lapse function N always equal to 1 in the regions inside and outside the bubble but with large values in the Natario warped region

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix A in this work)(see Appendix F in [12] with an adaption from Appendix J in [23])(This was also adapted from section 3 in [24])

$$ds^2 = (N^2 - 2NX_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \qquad (59)$$

The equation of the Natario vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \qquad (60)$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C in [12]):

$$X^t = 2n(rs)rscos\theta a \qquad (61)$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \qquad (62)$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \qquad (63)$$

The covariant shift vector components X_t,X_{rs} and X_θ are given by:

$$X_t = 2n(rs)rscos\theta a \qquad (64)$$

$$X_{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \qquad (65)$$

$$X_\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2\sin\theta \qquad (66)$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx + xdvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (N^2 - 2NX_t + X_t X^t - X_{rs} X^{rs})dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{67}$$

But since $X_t = X^t$ and $X_{rs} = X^{rs}$ the equation can be written as given below:

$$ds^2 = (N^2 - 2NX^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs} drs)dt - drs^2 \tag{68}$$

$$X^t = 2n(rs)rsa \tag{69}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{70}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{71}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero[5] and the covariant shift vector $X_{rs} = 2[2n(rs)^2]at$ with $X_{rs} = 0$ inside the bubble and $X_{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

A lapse function N with values different than 1 adapted to the Natario warp drive that obeys the $3+1$ ADM formalism with variable velocities must possess the following properties:

- inside the warp bubble(flat spacetime where the spaceship is located)the lapse function is equal to 1
- outside the warp bubble(flat spacetime where an external observer watches the ship passing by)the lapse function is also equal to 1
- in the Natario warped region(warp bubble walls curved spacetime) the lapse function must possesses a large value at least greater than or equal to the modulus of the ship velocity to keep the warp bubble totally connected.(see section 5 in [24])

[5] except in the neighborhoods of the bubble radius.See Section 2 in [12]

5 Horizons(causally disconnected portions of spacetime geometry in the equation of the Natario warp drive spacetime metric with a variable speed vs and a constant acceleration a in the original $1+1$ ADM formalism) with a lapse function N always equal to 1 inside and outside the Natario bubble but with large values in the Natario warped region

Like the section 4 in [24] the mathematical discussions of this section also uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.(Unlike the section 4 in [24] where from the geometrical point of view the photon stopped in the Horizon and the outermost layers of the bubble were causally disconnected from the observer in the center of the bubble for the case of constant velocity vs) in this section and still from a geometrical point of view we will demonstrate that in the case of variable velocities and large lapse functions the photon do not stops in the Horizon and the Horizon do not exists and the outermost layers of the bubble are causally connected to the observer in the center of the bubble.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime using a lapse function N:

$$ds^2 = (N^2 - 2NX^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{72}$$

The contravariant shift vector components X^t and X^{rs} are then:

$$X^t = 2n(rs)rsa \tag{73}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{74}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{75}$$

The term $N^2 - 2NX^t + (X^t)^2$ in the Natario warp drive equation for variable speed vs and constant acceleration a and a large lapse function N in a $1+1$ spacetime using a lapse function N can be simplified as:

$$N^2 - 2NX^t + (X^t)^2 = (N - (X^t))^2 \tag{76}$$

Hence the equation becomes:

$$ds^2 = ((N - (X^t))^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{77}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for variable speed vs and constant acceleration a and a large lapse function N in a $1+1$ spacetime we have:

$$0 = ((N - (X^t))^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{78}$$

Dividing both sides by dt^2 we have:

$$0 = ((N - (X^t))^2 - (X^{rs})^2) + 2(X^{rs}\frac{drs}{dt}) - (\frac{drs}{dt})^2 \tag{79}$$

Making the following algebraic substitution:

$$U = \frac{drs}{dt} \tag{80}$$

We have:

$$0 = ((N - (X^t))^2 - (X^{rs})^2) + 2(X^{rs}U) - (U)^2 \tag{81}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$(U)^2 - 2(X^{rs}U) - ((N - (X^t))^2 - (X^{rs})^2) = 0 \tag{82}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X^{rs}) \pm \sqrt{4(X^{rs})^2 + 4((N - (X^t))^2 - (X^{rs})^2)}}{2} \tag{83}$$

$$U = \frac{2(X^{rs}) \pm \sqrt{4(X^{rs})^2 + 4(N - (X^t))^2 - 4(X^{rs})^2}}{2} \tag{84}$$

The simplified algebraic expression becomes:

$$U = \frac{2(X^{rs}) \pm \sqrt{4(N - (X^t))^2}}{2} \tag{85}$$

Which leads to:

$$U = \frac{2(X^{rs}) \pm 2(N - (X^t))}{2} \tag{86}$$

And the final result is then given by:

$$U = X^{rs} \pm (N - (X^t)) \tag{87}$$

The above equation have two possible solutions U respectively $U = X^{rs} + (N - (X^t))$ and $U = X^{rs} - (N - (X^t))$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X^{rs} - (N - (X^t)) = X^{rs} + X^t - N \tag{88}$$

$$U_{rear} = X^{rs} + (N - (X^t)) = X^{rs} - X^t + N \tag{89}$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X^{rs} - (N - (X^t)) = X^{rs} + X^t - N \tag{90}$$

$$X^t = 2n(rs)rsa \tag{91}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at = 4n(rs)^2at + 2rsn'(rs)at = 2n(rs)2n(rs)at + 2rsn'(rs)at \tag{92}$$

Note that unlike the section 4 in [24] when we got only one contravariant shift vector component X^{rs} and it was this component that dictated the Horizon behavior of the front solution for the quadratic form now we get two contravariant shift vector components X^{rs} and X^t for the front solution of the quadratic form.This would be algebraically more complicated to be manipulated but fortunately we can write the component X^{rs} in function of the component X^t simplifying greatly the analysis.Using the following algebraic expressions both written in function of X^t

$$\frac{X^t}{rs} = 2n(rs)a \tag{93}$$

$$\frac{X^t}{n(rs)} = 2rsa \tag{94}$$

We can write X^{rs} in function of X^t as follows:

$$X^{rs} = 2n(rs)2n(rs)at + 2rsn'(rs)at = 2n(rs)\frac{X^t}{rs}t + n'(rs)\frac{X^t}{n(rs)}t \tag{95}$$

$$X^{rs} = 2\frac{n(rs)}{rs}tX^t + \frac{n'(rs)}{n(rs)}tX^t \tag{96}$$

Simplifying we get:

$$X^{rs} = tX^t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] \tag{97}$$

And the final solution of the quadratic form for the photon sent to the front part of the bubble is finally given by:

$$U_{front} = X^{rs} + X^t - N = tX^t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + X^t - N \tag{98}$$

The expression simplified leads to:

$$U_{front} = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N \tag{99}$$

$$U_{front} = (2n(rs)rsa)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N \tag{100}$$

The final solution of the quadratic form for the photon sent to the front part of the bubble is:

$$U_{front} = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N \tag{101}$$

$$U_{front} = (2n(rs)rsa)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N \tag{102}$$

Again note that unlike the section 4 in [24] where the dominant term was the component X^{rs} now the dominant term is the component X^t and $X^t = 2n(rs)rsa$.

Considering again a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) and assuming a continuous behavior for $n(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2n(rs)rsa$ from 0 to rsa we can clearly see that inside the bubble $2n(rs)rsa = 0$ because $n(rs) = 0$ and outside the bubble $2n(rs)rsa = rsa$ because $n(rs) = \frac{1}{2}$ and assuming also continuous values from 0 to rsa then somewhere in the Natario warped region where $0 < n(rs) < \frac{1}{2}$ we have the situation in which $2n(rs)rsa = 1$ because 1 lies in the continuous interval from 0 to rsa and in consequence $X^t = 1$.

The final solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ is:

$$U_{front}(X^t = 1) = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N = t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1 - N \tag{103}$$

Simplifying the result leads ourselves to:

$$U_{front}(X^t = 1) = t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1 - N \neq 0!!! \tag{104}$$

Note that unlike the section 4 in [24] the result is not zero !!!.The photon do not stops in the Natario warped region and the Horizon no longer exists!!!.

The place where $X^t = 2n(rs)rsa = 1$ is the place where the Natario shape function is $n(rs) = \frac{1}{2rsa}$ well inside the Natario warped region in which $0 < \frac{1}{2rsa} < \frac{1}{2}$ with $a >= 1$ and of course $rs > 0$.

Rewriting the solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ using the value of the Natario shape function $n(rs) = \frac{1}{2rsa}$ we get:

$$U_{front}(X^t = 1) = t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1 - N = t[2\frac{\frac{1}{2rsa}}{rs} + \frac{n'(rs)}{\frac{1}{2rsa}}] + 1 - N \neq 0!!! \tag{105}$$

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2 a} + 2n'(rs)rsa] + 1 - N \neq 0!!! \tag{106}$$

The solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ using the value of the Natario shape function $n(rs) = \frac{1}{2rsa}$ is:

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2a} + 2n'(rs)rsa] + 1 - N \neq 0!!! \tag{107}$$

Note that both the expressions $\frac{1}{rs^2a}$ and $2n'(rs)rsa$ have fractionary values close to zero but always greater than zero and above everything else not zero at all!!!.In the first expression we have both $rs > 0$ and $a \geq 1$ and in the second expression the derivative of the shape function must have low values in order to reduce the needs of negative energy density to sustain a warp bubble.

Then we can easily see that $0 < \frac{1}{rs^2a} < 1$ due to $rs > 0$ and $a \geq 1$ in the fraction and $0 < 2n'(rs)rsa < 1$ see section 3 in [11] for the low values of the square derivative of the Natario shape function able to reduce the negative energy density requirements implying in a low value for the derivative of the shape function.The expressions can be written as follows:

$$\frac{1}{rs^2a} \neq 0!!! \rightarrow \frac{1}{rs^2a} \simeq 0!!! \rightarrow \frac{1}{rs^2a} > 0!!! \rightarrow rs > 0 \rightarrow a \geq 1 \tag{108}$$

$$2n'(rs)rsa \neq 0!!! \rightarrow 2n'(rs)rsa \simeq 0!!! \rightarrow 2n'(rs)rsa > 0!!! \tag{109}$$

Then the solution of the quadratic form for the photon sent to the front part of the bubble when $X^t = 1$ using the value of the Natario shape function $n(rs) = \frac{1}{2rsa}$ is better written as:

$$U_{front}(X^t = 1) = t[\frac{1}{rs^2a} + 2n'(rs)rsa] + 1 - N \neq 00!!! \tag{110}$$

The result is not zero!!!.The Horizon do not exists!!!,

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for variable speed vs and constant acceleration a and a large lapse function N in a 1+1 spacetime.We know from the section 4 in [24] that in the case of the Natario warp drive with constant speed the negative energy density covers the entire bubble.(see Appendices B,C and D in [23]).Unfortunately we dont have the distribution of the negative energy density for the case of variable speeds and a large lapse function N .See section 5 pg 15 in [12] for the considerations of the negative energy density in the solution for variable speeds.Then we dont know if the negative energy density covers the entire bubble in the case of variable speeds and a large lapse function N but if this happens and since the negative energy density have repulsive gravitational behavior(see pg [116(a)][116(b)] in [13]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which would perhaps exists in the front of the bubble never reaching the bubble walls.

The solution to control the warp bubble would then be similar to the solution presented in the end of the section 4 in [24] using "pre-programmed" micro warp bubbles as described by pg 28 in [7] and pg 31 in [8] resembling the idea outlined in fig 7 pg [96(a)][83(b)] in [14].

Back again to the solution of the photon sent to the front of the bubble:

$$U_{front} = X^{rs} - (1 - (X^t)) = X^{rs} + X^t - N \tag{111}$$

This is the solution for a Natario warp drive metric with variable velocities and a lapse function.Compare with the solution of the photon sent to the front of the bubble in a Natario warp drive metric with fixed velocity without lapse function given in section 4 in [24]:

$$U_{front} = X^{rs} - 1 \tag{112}$$

The term in X^t and the lapse function affects the whole structure of the spacetime geometry eliminating once for all the problem of the Horizon. When the velocity is constant and without the lapse function the term in X^t vanishes and $N = 1$ leaving only the term is term in X^{rs} and in consequence $X^{rs} - 1$ and as already seen in section 4 in [24] the Horizon appears.

6 Conclusion:

In this work we presented the new equations for the warp drive spacetime according to Natario with variable velocity vs and constant acceleration a in the $3+1$ ADM formalism:

- 1)-equation without the lapse function

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{113}$$

- 2)-equation with the lapse function N

$$ds^2 = (N^2 - 2NX_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{114}$$

Some very important things both equations have in common:

- 1)- Both equations satisfies the Natario definition and condition for a warp drive spacetime using the same Natario shape function $n(rs)$ which gives 0 inside the bubble $\frac{1}{2}$ outside the bubble and $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.
- 2)- The same Natario shape function $n(rs)$ appears in the contravariant and covariant components of both Natario vectors.
- 3)- The same Natario shape function $n(rs)$ appears in the definition of the equation of the variable velocity $vs = 2n(rs)at$

A real and fully functional warp drive must encompasses accelerations or de-accelerations in order to go from 0 to 200 times light speed or even faster in the beginning of an interstellar journey and to slow down to 0 again in the end of the interstellar journey.

Both the Alcubierre and Natario original geometries encompasses warp drives of constant velocities so we presented an expanded version of the Natario vector in order to encompass time coordinate as a new Canonical Basis for the Hodge Star generating an extended version of the original Natario warp drive equation which of course encompasses accelerations or de-accelerations and variable velocities.

An accelerated warp drive can only exists if the astronaut in the center of the warp bubble can somehow communicate with the warp bubble walls sending instructions to change its speed.But for signals at light speed the Horizon exists at least for the Natario warp drive with constant velocity so light speed cannot be used to send signals to the front of the bubble in this case of fixed velocities.

In this work we analyzed the behavior of photons being sent to the front of the warp bubble by an observer in the center of the bubble using the null-like geodesics of General Relativity $ds^2 = 0$ in both the Natario warp drive metrics with or without the lapse function and variable speeds in the simplified case of the $1+1$ ADM formalism.

We used quadratic equations to analyze the behavior of photons being sent to the front of the warp bubble.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons and we arrived at the following results:

- 1)-Natario warp drive metric with variable velocity and without the lapse function:

Unlike the cases of section 4 in [20] or section 4 in [24] where the dominant term in the solution of the quadratic form was the contravariant spatial component X^{rs} now the dominant term is the contravariant time component $X^t = 2n(rs)rsa$ and the solution or the root of the quadratic form is given by:

$$U_{front} = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 \tag{115}$$

$$U_{front} = (2n(rs)rsa)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - 1 \tag{116}$$

Note that when $X^t = 2n(rs)rsa = 1$ the final result is not zero the photon do not stops and the Horizon do not exists.

- 1)-Natario warp drive metric with variable velocity and with the lapse function N:

$$U_{front} = (X^t)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N \tag{117}$$

$$U_{front} = (2n(rs)rsa)(t[2\frac{n(rs)}{rs} + \frac{n'(rs)}{n(rs)}] + 1) - N \tag{118}$$

Note that when $X^t = 2n(rs)rsa = 1$ the final result is not zero the photon do not stops and also the Horizon do not exists.

The term in X^t and the lapse function affects the whole structure of the spacetime geometry eliminating once for all the problem of the Horizon.

These solutions presented for the Natario warp drive spacetime are finally totally connected!!!

The Natario warp drive spacetime is a very rich environment to study the superluminal features of General Relativity because now we have two viable spacetime metrics totally connected and not only one and the geometry of the new equations in the $3+1$ spacetime is still unknown and needs to be cartographed.

Because collisions between the walls of the warp bubble and the hazardous particles of the Interstellar Medium(IM) would certainly occurs in a real superluminal interstellar spaceflight we borrowed the idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection of the spaceship and the crew members in the Natario warp drive equation for constant speed vs(see pg 46 in [11]).

Our idea was to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the dangerous IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

A submicroscopic outer radius of the bubble being the only part in contact with our Universe would mean a submicroscopic surface exposed to the collisions against the hazardous IM particles thereby reducing the probabilities of dangerous impacts against large objects (comets asteroids etc) enhancing the protection level of the spaceship and hence the survivability of the crew members.

Any future development for the Natario warp drive must encompass the more than welcome idea of Chris Van Den Broeck and this idea can also be easily implemented in the Natario warp drive with variable velocity with or without lapse functions.Since the Broeck idea is independent of the Natario geometry wether the lapse function is present or not we did not covered the Broeck idea here because it was already covered in [11] and in order to discuss the geometry of a Natario warp drive with variable velocity and with or without lapse functions the Broeck idea is not needed here however the Broeck idea must appear in a real Natario warp drive with variable velocity vs and with or without lapse functions concerning realistic superluminal interstellar spaceflights.

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.We also dont know how to generate the lapse function either.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that due to the two totally connected solutions presented here the Natario-Broeck warp drive will certainly survive the passage of the Century XXI and will arrive to the Future.The Natario-Broeck warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

7 Appendix A:mathematical demonstration of the Natario warp drive equation for a variable speed vs and a constant acceleration a in the original $3+1$ ADM Formalism according to MTW and Alcubierre using a lapse function N

In the Appendix C in [12] we defined a variable bubble velocity vs due to a constant acceleration a as follows:

$$vs = 2n(rs)at \tag{119}$$

And we obtained the Natario vector nX for a Natario warp drive with variable velocities defined as follows:

$$nX = vs(2n(rs)\ cos\theta e_r - [2n(rs) + rsn'(rs)]\sin\theta e_\theta) + rscos\theta(2[atn'(rs)e_r + n(rs)ae_t]) \tag{120}$$

$$nX = 2n(rs)at(2n(rs)\ cos\theta e_r - [2n(rs) + rsn'(rs)]\sin\theta e_\theta) + rscos\theta(2[atn'(rs)e_r + n(rs)ae_t]) \tag{121}$$

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{122}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rsd\theta \tag{123}$$

Remember that $x = rscos\theta$(see pg 5 in [2]). Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) we can see that the Natario vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * d(vs)$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2]).Working with some algebra we got:

$$nX = 2n(rs)rscos\theta ae_t + 2[2n(rs)^2 + rsn'(rs)]atcos\theta e_r - 2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta e_\theta \tag{124}$$

$$nX = 2n(rs)rscos\theta adt + 2[2n(rs)^2 + rsn'(rs)]atcos\theta drs - 2n(rs)at[2n(rs) + rsn'(rs)]rs\sin\theta d\theta \tag{125}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs)rscos\theta a \tag{126}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{127}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{128}$$

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg $[65(b)]$ $[80(a)]$ in [16].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg $[65(b)]$ $[80(a)]$ in [16])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg $[66(b)]$ $[81(a)]$ in [16])
(see also fig 21.2 pg $[506(b)]$ $[533(a)]$ in [15] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg $[507(b)]$ $[534(a)]$ in [15])[6]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij} dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [15]
with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [16] using the signature $(-, +, +, +)$ we get the original equations of the $3 + 1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{129}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{130}$$

[6] we adopt the Alcubierre notation here

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{131}$$

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = \gamma_{tt}(N + \beta^t)^2 = \gamma_{tt}(N^2 + 2N\beta^t + \beta^t\beta^t) = (\gamma_{tt}N^2 + 2\gamma_{tt}N\beta^t + \gamma_{tt}\beta^t\beta^t) \tag{132}$$

$$\beta_t = \gamma_{tt}\beta^t \tag{133}$$

Remember that here we are working with geometrized units in which $c = 1$ so $\gamma_{tt} = 1$

$$\alpha^2 = (N^2 + 2N\beta_t + \beta_t\beta^t) \tag{134}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\gamma_{tt}(N + \beta^t)^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{135}$$

Since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{136}$$

$$ds^2 = -\gamma_{tt}(N + \beta^t)^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{137}$$

From the Appendix E in [12] we can write the $3+1$ metric as:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{138}$$

Note that the expression above is exactly the eq (2.2.4) pgs [67(b)] [82(a)] in [16].It also appears as eq 1 pg 3 in [1].Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{139}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{140}$$

$$ds^2 = (N^2 + 2N\beta_t + \beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{141}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{142}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} N^2 + 2N\beta_t + \beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{143}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$ and we modified the equation to insert the terms due to the lapse function α^2.(pg 2 in [2])

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{144}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{145}$$

Comparing all these equations

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{146}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{147}$$

$$ds^2 = \alpha^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{148}$$

$$\alpha^2 = \gamma_{tt}(N + \beta^t)^2 \tag{149}$$

$$\alpha^2 = (N^2 + 2N\beta_t + \beta_t\beta^t) \tag{150}$$

$$ds^2 = \gamma_{tt}(N + \beta^t)^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{151}$$

$$ds^2 = (N^2 + 2N\beta_t + \beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{152}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} N^2 + 2N\beta_t + \beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{153}$$

With these

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{154}$$

$$ds^2 = \gamma_{tt}(N - X^t)^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{155}$$

$$\alpha^2 = \gamma_{tt}(N - X^t)^2 = \gamma_{tt}(N^2 - 2NX^t + X^tX^t) = (\gamma_{tt}N^2 - 2\gamma_{tt}NX^t + \gamma_{tt}X^tX^t) = (N^2 - 2NX_t + X_tX^t) \tag{156}$$

The generic equations for the Natario warp drive that obeys the $3+1$ ADM formalism are given below:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{157}$$

$$ds^2 = \gamma_{tt}(N - X^t)^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{158}$$

$$\alpha^2 = \gamma_{tt}(N - X^t)^2 = \gamma_{tt}(N^2 - 2NX^t + X^t X^t) = (\gamma_{tt}N^2 - 2\gamma_{tt}NX^t + \gamma_{tt}X^t X^t) = (N^2 - 2NX_t + X_t X^t) \tag{159}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector both for the spatial components of the Natario vector.In the same way we can see that $\beta^t = -X^t$,$\beta_t = -X_t$ and $\beta_t\beta^t = X_t X^t$ with X^t being the contravariant form of the Natario shift vector and X_t being the covariant form of the Natario shift vector for the time component of the Natario vector.Hence we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{160}$$

$$ds^2 = (N^2 - 2NX_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{161}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{162}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} N^2 - 2NX_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{163}$$

Looking to the equation of the Natario vector nX:

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{164}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{165}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector with a constant acceleration a are defined by:

$$X^t = 2n(rs)rs cos\theta a \tag{166}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at cos\theta \tag{167}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{168}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Remember also that $\gamma_{tt} = 1$.Then the covariant shift vector components X_t,X_{rs} and X_θ with $r = rs$ are given by:

$$X_t = \gamma_{tt} X^t \tag{169}$$

$$X_i = \gamma_{ii} X^i \tag{170}$$

$$X_t = \gamma_{tt} X^t = 2n(rs) rs cos\theta a \tag{171}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = X^r = X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{172}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = rs^2 X^\theta = X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2 \sin\theta \tag{173}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (N^2 - 2NX_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{174}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} N^2 - 2NX_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{175}$$

Then the equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration a in the original $3+1$ ADM formalism using a lapse function N is given by:

$$ds^2 = (N^2 - 2NX_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{176}$$

$$ds^2 = (N^2 - 2NX_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{177}$$

In the Appendix F in [12] we defined $\alpha^2 = \gamma_{tt}(1 + \beta^t)^2$ however to include the lapse function N the better definition is $\alpha^2 = \gamma_{tt}(N + \beta^t)^2$.

The lapse function N have the following properties:

- inside the warp bubble(flat spacetime where the spaceship is located)the lapse function is equal to 1
- outside the warp bubble(flat spacetime where an external observer watches the ship passing by)the lapse function is also equal to 1
- in the Natario warped region(warp bubble walls curved spacetime) the lapse function must possesses a large value at least greater than or equal to the modulus of the ship velocity to keep the warp bubble totally connected.(see section 5 in [24])

8 Remarks

References [4],[13],[14],[15],[16], [17] and [18] are standard textbooks used to study General Relativity and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [15] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader

The number of pages mentioned in the bibliographic references stored as e-prints in arXiv or HAL is the number of the page displayed in the bottom line of the Adobe PDF Acrobat Reader

9 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[7]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[8][9]

[7] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[8] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[9] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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