Simple, Mysterious, New Type and Best Possible Integral Inequalities

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February 27, 2024

Abstract: In this note, we give simple, mysterious, new type and best possible integral inequalities.

1 Introduction and restriction of reproducing kernel Hilbert spaces

In order to show the basic background of this note, we recall the restriction and extension of reproducing kernel Hilbert spaces.

We consider a positive definite quadratic form function (reproducing kernel) $K : E \times E \to \mathbb{C}$. We consider restriction of K to $E_0 \times E_0$, where E_0 is a subset of E. Of course, the restriction is again a positive definite quadratic form function on the subset $E_0 \times E_0$. We shall consider the relation between the two reproducing kernel Hilbert spaces.

Theorem A. ([6], pages 78-80). *Let E_0 be a subset of E. Then the Hilbert space that $K|E_0 \times E_0 : E_0 \times E_0 \to \mathbb{C}$ defines is given by:*

$$H_{K|E_0\times E_0}(E_0) = \{f \in \mathcal{F}(E_0) \,:\, f = \tilde{f}|E_0 \text{ for some } \tilde{f} \in H_K(E)\}. \tag{1.1}$$

Furthermore, the norm is expressed in terms of the one of $H_K(E)$:

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} = \min\{\|\tilde{f}\|_{H_K(E)} \,:\, \tilde{f} \in H_K(E),\, f = \tilde{f}|E_0\}. \tag{1.2}$$

In Theorem A, note that the inequality, for any function $f \in H_K(E)$

$$\|f\|_{H_{K|E_0 \times E_0}(E_0)} \le \|f\|_{H_K(E)}, \tag{1.3}$$

that is, the restriction map is a bounded linear operator.

In addition, for the minimum extension formula we have the general formula in Theorem A,

$$f(p) = (f|E_0(\cdot), K(\cdot, p))_{H_{K|E_0 \times E_0}(E_0)},$$

for the minimum norm extension f of $f|E_0$. See the proof of Proposition 2.5 in [6] (pages 79-80), in particular, (2.48).

With these strong motivations, we gave the realization of restricted reproducing kernels in [7] for some Sobolev Hilbert spaces.

The space $H_S(\mathbb{R})$ is comprising of absolutely continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_{H_S(\mathbb{R})} \equiv \sqrt{\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2) dx}. \tag{1.4}$$

The Hilbert space $H_S(\mathbb{R})$ admits the reproducing kernel (Green function)

$$K(x, y) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1 + \xi^2} \exp(i(x - y)\xi) d\xi = \frac{1}{2} e^{-|x-y|} \quad (x, y \in \mathbb{R}). \tag{1.5}$$

Its restriction to the closed interval $[a, b]$ is the reproducing kernel Hilbert space $H_S[a, b] = W^{1,2}[a, b]$ as a set of functions, and the norm is given by

$$\|f\|_{H_S[a,b]} \equiv \sqrt{\left(\int_a^b (|f(x)|^2 + |f'(x)|^2) \, dx \right) + |f(a)|^2 + |f(b)|^2}. \tag{1.6}$$

([6], pages 10-16).

In particular, we obtain the best possible inequality:

$$\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2) dx \ge \left(\int_a^b (|f(x)|^2 + |f'(x)|^2) \, dx \right) + |f(a)|^2 + |f(b)|^2. \tag{1.7}$$

We obtained several realizations of restricted reproducing kernel Hilbert spaces as in (1.6), however, they are, in general, involved. See [4], [6]. The formula (1.6) is a simple result, however, the realization of the restricted reproducing kernel spaces is, in general, complicated in this sense.

For the Sobolev Hilbert space $W^{2,2}(\mathbb{R})$ defined to be the completion of $C_{\mathrm{c}}^{\infty}(\mathbb{R})$ with respect to the norm:

$$\|f\|_{W^{2,2}(\mathbb{R})} = \sqrt{\|f''\|_{L^2(\mathbb{R})}{}^2 + 2\|f'\|_{L^2(\mathbb{R})}{}^2 + \|f\|_{L^2(\mathbb{R})}{}^2},$$

we have the reproducing kernel

$$G(s,t) \equiv \frac{1}{4} e^{-|s-t|}(1+|s-t|) \quad (s,t \in \mathbb{R})$$

([6], pages 21-22).

For simplicity, we shall consider functions in real valued functions.

We looked the reproducing kernel Hilbert space $W_S([a,b]), (a<b)$ admitting the restricted reproducing kernel $G(s,t)$ to the interval $[a,b]$:

$$\begin{aligned}\|f\|^2_{W_S([a,b])} &= \|f''\|^2_{L^2([a,b])} + 2\|f'\|^2_{L^2([a,b])} + \|f\|^2_{L^2([a,b])} \\ &+2(f(a)^2 - f(a)f'(a) + f'(a)^2) + 2(f(b)^2 - f(b)f'(b) + f'(b)^2).\end{aligned} \tag{1.8}$$

Let

$$K(s,t) \equiv \int_0^\infty \frac{\cos(s\,u)\cos(t\,u)}{u^2+1}\,du = \frac{\pi}{4}\left(\exp(-|s-t|) + \exp(-s-t)\right) \tag{1.9}$$

for $s,t>0$. Then $H_K(0,\infty) = W^{1,2}(0,\infty)$ as a set of functions and the norm is given by:

$$\|f\|_{H_K(0,\infty)} = \sqrt{\frac{2}{\pi}\int_0^\infty (|f'(u)|^2 + |f(u)|^2)\,du} \tag{1.10}$$

([6], pages 12-13). From the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the realization of the norm

$$\|f\|^2_{H_K[a,b]} = \frac{2}{\pi}\frac{1-\exp(-2a)}{1+\exp(-2a)}|f(a)|^2 \tag{1.11}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du+\frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t)\equiv\int_0^\infty \frac{\sin(s\,u)\sin(t\,u)}{u^2+1}\,du=\frac{\pi}{4}\left(\exp(-|s-t|)-\exp(-s-t)\right)$$

for $s,t>0$.

Then we have

$$H_K(0,\infty)=\{f\in \mathrm{AC}(0,\infty)\,:\,f(0)=0\} \tag{1.12}$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)}=\sqrt{\frac{2}{\pi}\int_0^\infty (|f'(u)|^2+|f(u)|^2)\,du} \tag{1.13}$$

([6], pages 13-14). For the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the realization of the norm

$$\|f\|^2_{H_K[a,b]}=\frac{2}{\pi}\frac{1+\exp(-2a)}{1-\exp(-2a)}|f(a)|^2 \tag{1.14}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du+\frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t)\equiv\min(s,t)\quad(s,t>0). \tag{1.15}$$

Then we have

$$H_K(0,\infty)=\left\{f\in W^{1,2}(0,\infty)\,:\,\lim_{\varepsilon\downarrow 0}f(\varepsilon)=0\right\} \tag{1.16}$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)}=\sqrt{\int_0^\infty |f'(u)|^2\,du} \tag{1.17}$$

([6], pages 14-15). For the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the realization of the norm

$$\|f\|^2_{H_K[a,b]} = \frac{1}{a}|f(a)|^2 + \int_a^b |f'(u)|^2\, du. \tag{1.18}$$

We have many type Sobolev Hilbert spaces. For example,
for $\omega^2 = \gamma^2 - \alpha^2 > 0$, the kernel

$$K(s,t) = \frac{\exp(-\alpha|s-t|)}{4\alpha\gamma^2} \cos(\omega|s-t|) + \frac{\alpha}{\omega}\sin(\omega|s-t|)$$

is the reproducing kernel for the Sobolev Hilbert space admitting the norm

$$||u||^2 = 4\alpha\gamma^2 u(a)^2 + 4\alpha u'(a)^2$$

$$+ \int_a^b \left(u''(t) + 2\alpha^2 u'(t) + \gamma^2 u(t)\right)^2 dt$$

(E. Parzen, [2]).

See also [1] and the recent paper A. Yamada ([8]).

2 Results

From the above results, we obtain the simple best possible integral inequalities

Theorem.
For any function $f \in H_K[a,b], a > 0$ in (1.18), we have

$$\int_a^b |f'(x)|^2\, dx \geq \frac{1}{b}|f(b)|^2 - \frac{1}{a}|f(a)|^2. \tag{2.1}$$

For any function $f \in H_K[a,b]$ in (1.6), we have

$$\int_a^b \left(|f'(x)|^2 + |f(x)|^2\right)\, dx \geq |f(b)|^2 - |f(a)|^2. \tag{2.2}$$

For any function $f \in H_K[a,b], a > 0$ in (1.11) or in (1.13), we have

$$\int_a^b (|f'(x)|^2 + |f(x)|^2)\, dx \geq \frac{1+\exp(-2b)}{1-\exp(-2b)}|f(b)|^2 - \frac{1+\exp(-2a)}{1-\exp(-2a)}|f(a)|^2. \tag{2.3}$$

For any function $f \in H_K[a,b]$ in (1.8), we have

$$\int_a^b \left(|f''(x)|^2 + 2|f'(x)|^2 + |f(x)|^2\right) dx \tag{2.4}$$
$$\geq 2(f(b)^2 - f(b)f'(b) + f'(b)^2) - 2(f(a)^2 - f(a)f'(a) + f'(a)^2).$$

Indeed, for example, from (1.18), we have, for $a < b < c$, by considering the extension of $H_K[a,b]$ to $H_K[a,c]$

$$\|f\|^2_{H_K[a,c]} \geq \|f\|^2_{H_K[b,c]},$$

that is,

$$\frac{1}{a}|f(a)|^2 + \int_a^c |f'(x)|^2 \, dx.$$
$$\geq \frac{1}{b}|f(b)|^2 + \int_b^c |f'(x)|^2 \, dx$$

and the desired result.

Other results may be given similarly.

3 Conclusion and open problems

We obtained the new type fundamental norm inequalities for Sobolev Hilbert spaces, and we can apply the inequalities in this way, for example

$$\frac{1}{b}|f(b)|^2 \leq \frac{1}{a}|f(a)|^2 + \int_a^b |f'(x)|^2 \, dx.$$

We wonder: for these inequalities, does there exist some elementary proof? And some applications and generalizations.

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