On eikonal and black hole entropy

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We formulate the eikonal equation in (3+1)-dimensional spherically symmetric curved space-time using Clebsch variables. We assume that the mass is the mass of a black hole and it is related to a black hole entropy through its area.

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The static spherically symmetric curved space-time (gravitational field) described by the Schwarzschild metric[1], an interval, can be written as

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(1-\frac{r_s}{r}\right)^{-1} dr^2 + r^2\ d\theta^2 + r^2 \sin^2\theta\ d\phi \quad (1)$$

where r is the spatial (radial) coordinate, the distance from the centre of a mass of the massive body (such as a black hole, a massive star) to the points (these points have the same distance from the centre) where the Schwarzschid metric is being evaluated, and

$$r_s = \frac{2GM}{c^2} \quad (2)$$

is the Schwarzschild radius, M is the mass of the spherically symmetric body, G is the gravitational constant, and c is the speed of light in flat space-time. Equation (1) is also known as the Schwarzschild solution[2] (to Einstein field equation). The Schwarzschild solution holds outside the surface of the massive body that is producing the gravitational field, where there is no matter[2].

In our previous work[3], we found that the eikonal equation in (3+1)-dimensional spherically symmetric curved space-time can be written below

$$\left|\partial_\mu \left\{\frac{c}{f_\theta}\int f\ \partial_\nu q\ f^*\ dx^\nu + ct\right\}\right| = \left(1-\frac{2GM}{c^2 r}\right)^{-1} \quad (3)$$

where $f\ \partial_\nu q = \vec{A}_\nu$ is the $U(1)$ gauge potential, f, q are Clebsch variables, f^* is the conjugate complex of f, and f_θ is angular frequency. Eq.(3) shows explicitly the relation between the mass of the spherically symmetric body that is producing the spherically symmetric gravitational field and the $U(1)$ gauge potential which represents the existence of the light ray.

The relation between a black hole entropy and its area can be written as[4]

$$S_{bh} = \frac{\eta k A}{l_p^2} \quad (4)$$

where S_{bh} is the entropy of a black hole, A is the area of a black hole, k is the Boltzmann's constant, η is a constant number of order unity, l_p is the Planck length

$$l_p = \left(\frac{\hbar G}{c^3}\right)^{1/2} \quad (5)$$

$\hbar = h/2\pi$, h is the Planck constant.

The surface area of a spherically symmetric black hole can be written as

$$A = 4\pi r_s^2 \quad (6)$$

By substituting eq.(2) to (6) we obtain

$$A = 4\pi\left(\frac{2GM}{c^2}\right)^2 \quad (7)$$

By substituting eq.(7) into (4), we obtain

$$S_{bh} = \frac{4\pi\eta k}{l_p^2}\left(\frac{2GM}{c^2}\right)^2 \quad (8)$$

By substituting (5) into (8), we obtain

$$S_{bh} = \frac{16\pi\eta k}{\hbar c}\ GM^2 \quad (9)$$

or

$$M = \sqrt{\frac{\hbar c}{16\pi\eta k G}\ S_{bh}} \quad (10)$$

By substituting (10) into (2), we obtain

$$r_s = l_p\sqrt{\frac{S_{bh}}{4\pi\eta k}} \quad (11)$$

Eq.(11) relates the Schwarzschild radius to a black hole entropy.

If we replace eq.(2) with (11) then we can write eq.(3) as

$$\left|\partial_\mu \left\{\frac{c}{f_\theta}\int f\ \partial_\nu q\ f^*\ dx^\nu + ct\right\}\right| = \left(1-\frac{l_p}{r}\ \sqrt{\frac{S_{bh}}{4\pi\eta k}}\right)^{-1} \quad (12)$$

Eq.(12) is the eikonal equation that describes the light ray propagation in static spherically symmetric space-time with the distance r from the centre related to the entropy of a black hole. It relates the $U(1)$ gauge potential to the entropy of a black hole. Does it mean that the $U(1)$ gauge potential shares similar properties with the entropy?

If we take $r = r_s$ then eqs.(1), (12) become

$$ds^2 = \infty \tag{13}$$

and

$$\left|\partial_\mu \left\{ \frac{c}{f_\theta} \int f \ \partial_\nu q \ f^* \ dx^\nu + ct \right\}\right| = \infty \tag{14}$$

respectively. Eqs.(13), (14) show us that there exist the singularity.

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