A weak Galerkin finite element method for the incompressible viscous Magneto-hydrodynamic boundary value problems

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Abstract

In this paper, we have studied the weak Galerkin finite element method for the incompressible viscous Magneto-hydrodynamic(MHD) equations.

A weak Galerkin finite element methods are based on new concept called discrete weak gradient, discrete weak divergence and discrete weak rotation, which are expected to play an important role in numerical methods for magneto-hydrodynamic equation.

This article intends to provide a general framework for managing differential, divergence, rotation operators on generalized functions. With the proposed method, solving the magneto-hydrodynamic (MHD) equation is that the classical gradient, divergence, rotation operators are replaced by the discrete weak gradient, divergence, rotation and apply the Galerkin finite element method. It can be seen that the solution of the weak Galerkin finite element method is not only continuous function but also totally discontinuous function. For the proposed method, optimal order error estimates are established in various norms.

Keywords: Weak Galerkin finite element method, Incompressible viscous magneto-hydrodynamic equations, Discrete weak gradient, Discrete weak divergence, Discrete weak rotation.

1. Introduction

Magneto-hydrodynamic(MHD) equations which have been widely used in industry and engineering, such as liquid metal cooling of nuclear reactors, process metallurgy, simulated aluminum electrolysis cells and so on are composed of Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism.

These equations are concerned with the viscous, incompressible, electrically conducting fluid and an external magnetic field.

In this paper, we consider the magneto-hydrodynamic equations

$$\frac{\partial \mathrm{u}}{\partial t} - \gamma \Delta \mathrm{u} + (\mathrm{u}\nabla)\mathrm{u} + \nabla p = a \cdot rot\mathrm{B} \times \mathrm{B}, \quad x \in \Omega, \quad t \in (0,T) \tag{1.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + rot\mathbf{E} = 0, \quad \mathbf{j} = rot\mathbf{B} = \frac{1}{\gamma_\mu}(\mathbf{E} + \mathbf{u} \times \mathbf{B} + \mathbf{J}_c) \tag{1.2}$$

$$div\mathbf{u} = 0, \quad div\mathbf{B} = 0 \tag{1.3}$$

with the following initial condition and homogeneous Dirichlet boundary conditions

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0(x), \quad \mathbf{B}\big|_{t=0} = \mathbf{B}_0(x) \qquad x \in \Omega \tag{1.4}$$

$$\mathbf{u} = 0, \quad \mathbf{B}\cdot\mathbf{n} = 0, \quad \mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{J}_c \times \mathbf{n} = 0 \quad (x,t) \in \partial\Omega \times (0,T) \tag{1.5}$$

where $\Omega \subset R^n$ $(n = 2, 3)$ is a bounded-closed, convex domain with Lipschitz-continuous boundary $\partial\Omega$, $\mathbf{u}, \mathbf{B}, \mathbf{E}, \mathbf{j}$ are velocity, magnetic field, intensity of electric field, electric current density, $\mathbf{J}_c$

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is minor control possible electromotive force, p is hydrodynamic pressure, $\gamma = \frac{1}{R_e}$, $\gamma_\mu = \frac{1}{R_\mu}$, $a = \frac{M^2}{R_e R_\mu}$ and the three parameter R_e, R_μ, M represent hydrodynamic Reynolds number, magnetic Reynolds number and Hartmann number.

Because MHD equations have practical significance in many fields of science, technology and production, many researchers gave some research results for magneto-hydrodynamic. (see [1], [2], [7], [9])

Some research results are given as follows:

In [3–5], the mixed and stabilized FE methods were used to solve MHD equations.

Additionally, Y. He ([8]) studied an unconditional convergence of the Euler semi-implicit scheme and G. Yuksel and R. Ingram ([6]) investigated a full discretization of Crank–Nicolson scheme for the non-stationary MHD equations with small magnetic Reynolds numbers.

In [10], it was studied the design and analysis of some structure preserving finite element schemes for the magneto-hydrodynamics(MHD) system.

X. Feng et al ([11]) have applied some Uzawa-type iterative algorithms to the steady incompressible magneto-hydrodynamic(MHD) equations discretized by mixed finite element method.

In [12], it has focused on a fractional-step finite element method for the magneto-hydrodynamics problems in three-dimensional bounded domains.

In [13], the convergence analyses of standard Galerkin finite element method and a new highly efficient two-step algorithm for the stationary incompressible magneto–hydrodynamic equations was studied.

In [14], it was devoted to extension of boundary element method (BEM) for solving coupled equations in velocity and induced magnetic field for time dependent magneto-hydrodynamic (MHD) flows through a rectangular pipe.

In [15], Y. Rong and Y. Hou have studied a partitioned scheme based on Gauge-Uzawa finite element method for the 2D time-dependent incompressible magneto-hydrodynamics(MHD) equations.

In this paper, we are going to propose a formulation for the weak Galerkin finite element method for the magneto-hydrodynamic(MHD) equations (1.1) – (1.5).

The weak Galerkin(WG) method was recently introduced in [16] for second-order elliptic problems based on local RT elements.

It is an extension of the standard Galerkin finite element method where classical operators (e.g., gradient, divergence, and curl) are substituted by weakly defined operators.

Then, in [17], the weak Galerkin method was extended to allow arbitrary shapes of finite elements in a partition by adding parameter free stabilizer, which enforces a certain weak continuity and provides a convenient flexibility in mesh generation.

Through rigorous analysis, the optimal order of priori error estimates has been established for various weak Galerkin discretization schemes for second order elliptic equation in [16–18].

And the possibility of an optimal combination of polynomial spaces that minimizes the number of unknowns has been explored in several numerical experiments in [19].

On the base of the weak Galerkin mixed finite element methods, the weak Galerkin method for the Stokes equations was stated in [20].

In [28], it was considered the lowest-order weak Galerkin method for linear elasticity based on the displacement formulation.

Moreover, because the weak Galerkin method inherits the advantages and abandons the weaknesses of a discontinuous Galerkin or discontinuous Petrov–Galerkin method, it has been developed to solve many equations.

Thus, we refer to several papers for applications of the WG method to some other partial differential equations, such as, elliptic interface equations ([31]), Oseen equations ([21]), Helmholtz equations ([22, 23]), Darcy–Stokes equations ([24, 29, 30]), convection–diffusion–reaction equations ([32]), Sobolev equation ([33]) and parabolic equations ([25–27]) etc.

It is well known that the magneto-hydrodynamic equations involve a trilinear term, which changes the essence of all the problems considered so far (linear or nonlinear problems).

The goal of this article is to construct and analyze a stable weak Galerkin finite element method for the magneto-hydrodynamic equations (1.1) – (1.5) by using the definition of a weak trilinear term.

This method allows the use of finite element partitions with arbitrary shapes of polygons or polyhedra with shape regularity and parameter free.

An outline of this paper is as follows. In the next section, we introduce some notations for the magneto-hydrodynamic equations (1.1) – (1.5) and Sobolev spaces.

In Section 3, the fundamental definitions and weak Galerkin finite element scheme for the magneto-hydrodynamic equations are developed.

Then, in Section 4, we estimate the error of weak Galerkin finite element approximation solution for the incompressible viscous Magneto-hydrodynamic equations.

In Section 5, we give numerical experiment to verify the studied theoretical analysis. Finally, conclusions are drawn in Section 6.

2. Preliminary results and notations

From the systems of equations (1.1), (1.2)

$$\frac{\partial \mathbf{u}}{\partial t} - \gamma \cdot \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - a\cdot rot\mathbf{B}\times\mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \gamma_\mu \cdot rot\cdot rot\mathbf{B} - rot(\mathbf{u}\times\mathbf{B}) = rot\mathbf{J}_c$$

$$div\mathbf{u} = 0, \quad div\mathbf{B} = 0$$

Therefore, we consider the following non-stationary incompressible magneto-hydrodynamic equations

$$\frac{\partial \mathbf{u}}{\partial t} - \gamma \cdot \Delta \mathbf{u} + rot\mathbf{u}\times\mathbf{u} - a\cdot rot\mathbf{B}\times\mathbf{B} + \nabla(p + \frac{1}{2}|\mathbf{u}|^2) = 0 \quad (x,t)\in\Omega\times(0,T), \tag{2.1}$$

$$\frac{\partial \mathrm{B}}{\partial t} + \gamma_\mu rot\cdot rot\mathrm{B} - rot(\mathrm{u}\times\mathrm{B}) = rot\mathrm{J}_c \quad (x,t)\in\Omega\times(0,T), \tag{2.2}$$

$$div\mathbf{u} = 0, \quad div\mathbf{B} = 0 \quad (x,t)\in\Omega\times(0,T), \tag{2.3}$$

with the initial-boundary conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x), \quad \mathbf{B}|_{t=0} = \mathbf{B}_0(x) \qquad x\in\Omega \tag{2.4}$$

$$\mathbf{u} = 0, \quad \mathbf{B}\cdot\mathbf{n} = 0, \quad \mathbf{E}\times\mathbf{n} = 0, \quad \mathbf{J}_c\times\mathbf{n} = 0 \quad (x,t)\in\partial\Omega\times(0,T) \tag{2.5}$$

We introduce the following notations of some norms and spaces**:**

$$\mathbf{L}^2(\Omega) = [L^2(\Omega)]^d, \quad \mathbf{W}_2^1 = [W_2^1(\Omega)]^d, \quad (d = 2, 3)$$

$$\mathbf{H}(div, \Omega) = \{\mathbf{v} | \ \mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\cdot\mathbf{v}\in L^2(\Omega)\}$$

$$\mathbf{H}(rot, \Omega) = \{\mathbf{v} | \ \mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega)\}$$

$$\mathbf{H}_1 = \mathbf{H}_2 = \{\mathbf{w}\in\mathbf{L}^2(\Omega) | \ div\mathbf{w} = 0, \quad \mathbf{w}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

$$\mathbf{V}_1 = \{\mathbf{u}\in\mathbf{W}_2^1(\Omega) | \ div\mathbf{u} = 0, \ \mathbf{u}|_{\partial\Omega} = 0\},$$

$$\mathbf{V}_2 = \{\mathbf{B}\in\mathbf{W}_2^1(\Omega) | \ div\mathbf{B} = 0, \ \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

$$\mathbf{V}_3 = \mathbf{H}_0(rot, \ \Omega) = \{\mathbf{u}\in\mathbf{L}^2(\Omega) | \ rot\mathbf{u}\in\mathbf{L}^2(\Omega), \ \mathbf{u}\times\mathbf{n}|_{\partial\Omega} = 0\}$$

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\mathbf{x}, \qquad \|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}, \qquad \forall\mathbf{u}, \mathbf{v}\in\mathbf{H}_1 \text{ or } \mathbf{H}_2$$

$$[\mathbf{u}, \mathbf{v}] = (rot\mathbf{u}, \ rot\mathbf{v}) = \int_\Omega rot\mathbf{u}\cdot rot\mathbf{v} d\mathbf{x}, \ |\mathbf{u}| = \sqrt{[\mathbf{u}, \mathbf{u}]}, \quad \forall\mathbf{u}, \mathbf{v}\in\mathbf{V}_1 \text{ or } \mathbf{V}_2$$

$$\mathbf{H} = \mathbf{H}_1\times\mathbf{H}_2, \ \mathbf{V} = \mathbf{V}_1\times\mathbf{V}_2, \ \mathbf{V}\subset\mathbf{H} = \mathbf{H}'\subset\mathbf{V}'$$

where $(rot\mathbf{u}, \ rot\mathbf{v}) = (\nabla\mathbf{u}, \ \nabla\mathbf{v}) \ \ \forall\mathbf{u}, \mathbf{v}\in\mathbf{V}_1$ and $\mathbf{H}', \mathbf{V}'$ are dual spaces of $\mathbf{H}, \mathbf{V}$.

We define the operators $A_1:\mathbf{V}_1\to\mathbf{V}_1$, $A_2:\mathbf{V}_2\to\mathbf{V}_2$, $A:\mathbf{V}\to\mathbf{V}'$, $G:\mathbf{V}\times\mathbf{V}\to\mathbf{V}'$, $F:\mathbf{L}^2(\Omega)\to\mathbf{V}'$ following as:

$$
\begin{aligned}
&(A\mathbf{y},\mathbf{z})=\gamma(A_1\mathbf{u},\mathbf{v})+\gamma_\mu(A_2\mathbf{B},\mathbf{w})=a_1(\mathbf{u},\mathbf{v})+a_2(\mathbf{B},\mathbf{w})\\
&a_1(\mathbf{u},\mathbf{v})=\gamma(\nabla\mathbf{u},\nabla\mathbf{v})\\
&a_2(\mathbf{B},\mathbf{w})=\gamma_\mu(rot\mathbf{B},rot\mathbf{w})\\
&(G(\mathbf{y}_1,\mathbf{y}_2),\mathbf{z})=(rot\mathbf{u}_1\times\mathbf{u}_2-rot\mathbf{B}_2\times\mathbf{B}_1,\mathbf{v})-(\mathbf{u}_2\times\mathbf{B}_1,rot\mathbf{w})\\
&(F(\mathbf{J}_c),\mathbf{z})=(\mathbf{J}_c,rot\mathbf{w}),
\end{aligned}
$$

where $\mathbf{y}=\{\mathbf{u},\mathbf{B}\}$, $\mathbf{y}_1=\{\mathbf{u}_1,\mathbf{B}_1\}$, $\mathbf{y}_2=\{\mathbf{u}_2,\mathbf{B}_2\}$, $\mathbf{z}=\{\mathbf{v},\mathbf{w}\}\in\mathbf{V}$, $\mathbf{J}_c\in\mathbf{L}^2(\Omega)$

For the operator $A:\mathbf{V}\to\mathbf{V}'$,

$$
\begin{aligned}
&(A\mathbf{y},\mathbf{y})\ge\alpha|\mathbf{y}|^2,\quad \alpha=\min\{\gamma,\gamma_\mu\}\\
&(A\mathbf{y},\mathbf{z})=(\mathbf{y},A\mathbf{z}),\quad \forall\mathbf{y},\mathbf{z}\in\mathbf{V}
\end{aligned}
$$

For the operator $G:\mathbf{V}\times\mathbf{V}\to\mathbf{V}'$, representing $G(\mathbf{y},\mathbf{y})=G[\mathbf{y}]$,

$$(G(\mathbf{y},\mathbf{z}),\mathbf{z})=0,\ (G[\mathbf{y}],\mathbf{z})=(rot\mathbf{u}\times\mathbf{u},\mathbf{v})-(rot\mathbf{B}\times\mathbf{B},\mathbf{v})-(\mathbf{u}\times\mathbf{B},rot\mathbf{w}).$$

By the Hölder's inequality, we can know that norm satisfies $|\mathbf{u}|_{\mathbf{L}^4(\Omega)}\le K\|\mathbf{u}\|^{\frac{1}{4}}\cdot|\mathbf{u}|^{\frac{3}{4}}$.

For operator $G:\mathbf{V}\times\mathbf{V}\to\mathbf{V}'$, the trilinear form $(G(\mathbf{y}_1,\mathbf{y}_2),\mathbf{y}_3)$ satisfies

$$(G(\mathbf{y}_1,\mathbf{y}_2),\mathbf{y}_3)\le C_1|\mathbf{y}_1|^{\frac{3}{4}}\cdot\|\mathbf{y}_1\|^{\frac{1}{4}}\cdot|\mathbf{y}_2|\cdot|\mathbf{y}_3|^{\frac{3}{4}}\cdot\|\mathbf{y}_3\|^{\frac{1}{4}},\quad \mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\in V,\tag{2.6}$$

$$(G(\mathbf{y}_1,\mathbf{y}_2),\mathbf{y}_3)\le C_1|\mathbf{y}_1|\cdot|\mathbf{y}_2|^{\frac{1}{2}}\cdot\|A\mathbf{y}_2\|^{\frac{1}{2}}\cdot\|\mathbf{y}_3\|,\quad \mathbf{y}_1\in V,\ \mathbf{y}_2\in D(A),\ \mathbf{y}_3\in H,\tag{2.7}$$

where $C_1>0$ is a constant independent of Ω, R_e, R_μ. ([34])

The weak formulation of problem (2.1)-(2.5) can be written a variation forms as follows

$$
\begin{aligned}
&(\dot{\mathbf{y}},\mathbf{z})+(A\mathbf{y},\mathbf{z})+(G[\mathbf{y}],\mathbf{z})=(F(\mathbf{J}_c),\mathbf{z})\qquad \forall\mathbf{z}\in\mathbf{V},\ \forall t\in(0,T), &(2.8)\\
&\mathbf{y}(0)=\mathbf{y}_0\qquad \forall\mathbf{z}\in\mathbf{V}, &(2.9)
\end{aligned}
$$

where $\mathbf{y}=\{\mathbf{u},\mathbf{B}\}$ is called a weak solution if $\mathbf{y}\in L^2(0,T;\mathbf{V})$ and $\dot{\mathbf{y}}\in L^2(0,T;\mathbf{V}')$ are the solutions of equations (2.8)-(2.9).

Next, we will introduce the weak gradient operator, weak divergence operator and newly weak rotation operator defined on a space of generalized functions.

To explain weak gradient, weak divergence and weak rotation, let K be any polygonal domain with interior K^0 and boundary ∂K.

A weak function on the region K refers to a function $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}$ such that $\mathbf{v}_0\in\mathbf{L}^2(K)$ and $\mathbf{v}_b\in\mathbf{H}^{\frac{1}{2}}(\partial K)=\left[H^{\frac{1}{2}}(\partial K)\right]^d$, where the first component $\mathbf{v}_0$ can be understood as the value of $\mathbf{v}$ in t he interior of K and the second component $\mathbf{v}_b$ is the value of $\mathbf{v}$ on the boundary of K.

Denote by $\mathbf{W}(K)$ the space of weak functions associated with K; i.e.,

$$\mathbf{W}(K)=\left\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\middle|\ \mathbf{v}_0\in\mathbf{L}^2(K),\mathbf{v}_b\in\mathbf{H}^{\frac{1}{2}}(\partial K)\right\}\tag{2.10}$$

The dual of $\mathbf{L}^2(K)$ can be identified with itself by using the standard $\mathbf{L}^2$ inner product as the action of linear functional.

Definition 1.([16]) For any $\mathbf{v}\in\mathbf{W}(K)$, the weak gradient of $\mathbf{v}$ is defined as a linear function $\nabla_w\mathbf{v}$ in the dual space of $\mathbf{H}(div,K)$ whose action on each $\mathbf{q}\in\mathbf{H}(div,K)$ is given by

$$(\nabla_w\mathbf{v},\mathbf{q})_K=-(\mathbf{v}_0,\nabla\cdot\mathbf{q})_K+<\mathbf{v}_b,\mathbf{q}\cdot\mathbf{n}>_{\partial K},$$

where $\mathbf{n}$ is the outward normal direction to ∂K.

The discrete weak gradient operator denoted by $\nabla_{w,r,K}$ is defined as the unique polynomial $\nabla_{w,r,K}\mathbf{v}\in[P_r(K)]^{d\times d}$ satisfying the following equation:

$$(\nabla_{w,r,K}\mathbf{v},\mathbf{w})_K=-(\mathbf{v}_0,div\mathbf{w})_K+<\mathbf{v}_b,\mathbf{w}\cdot\mathbf{n}>_{\partial K},\quad\forall\mathbf{w}\in[P_r(K)]^{d\times d}\tag{2.11}$$

Definition 2.([16]). For any $\mathbf{v}\in\mathbf{W}(K)$, the weak divergence of $\mathbf{v}$ is defined as a linear function $div_w\mathbf{v}$ in the dual space of $W_2^1(K)$ whose action on each $q\in W_2^1(K)$ is given by

$$(div_w\mathbf{v},q)_K=-(\mathbf{v}_0,\nabla q)_K+<\mathbf{v}_b\cdot\mathbf{n},q>_{\partial K},$$

where $\mathbf{n}$ is the outward normal direction to ∂K.

The discrete weak divergence operator denoted by $div_{w,r,K}$ is defined as the unique polynomial $div_{w,r,K}\mathbf{v}\in P_r(K)$ satisfying the following equation:

$$(div_{w,r,K}\cdot\mathbf{v},q)_K=-(\mathbf{v}_0,\nabla q)_K+<\mathbf{v}_b\cdot\mathbf{n},q>_{\partial K},\quad\forall q\in P_r(K)\tag{2.12}$$

Then we introduce the definition of a weak rotation operator.

Definition 3. For any $\mathbf{v}\in\mathbf{W}(K)$, the weak rotation of $\mathbf{v}$ is defined as a linear function $rot_w\mathbf{v}$ in the dual space of $\mathbf{H}(rot,K)$ whose action on each $\mathbf{q}\in\mathbf{H}(rot,K)$ is given by

$$(rot_w\mathbf{v},\mathbf{q})_K=(\mathbf{v}_0,rot\mathbf{q})_K-<\mathbf{v}_b\times\mathbf{n},\mathbf{q}>_{\partial K},$$

where $\mathbf{n}$ is the outward normal direction to ∂K.

The discrete weak rotation operator denoted by $rot_{w,r,K}$ is defined as the unique polynomial $rot_{w,r,K}\in[P_r(K)]^d$ satisfying the following equation:

$$(rot_{w,r,K},\mathbf{q})_K=(\mathbf{v}_0,rot\mathbf{q})_K-<\mathbf{v}_b\times\mathbf{n},\mathbf{q}>_{\partial K},\quad\forall\mathbf{q}\in[P_r(K)]^d\tag{2.13}$$

3. Weak Galerkin finite element method (WGFEM)

In this section, we design a continuous time and discontinuous time WGFEM for the problem (2.8)-(2.9).

Let K_h be a regular, quasi-uniform partition of the domain Ω and $K\in K_h$ be any polygonal domain with interior K^0 and boundary ∂K ,where the mesh size $h=\max h_K$, h_K is the diameter of element K.

Then, we can introduce the discrete weak Galerkin finite element spaces on a given mesh: for the velocity variable,

$$\mathbf{V}_{1h}=\left\{\mathbf{u}_h=\{\mathbf{u}_0^h,\mathbf{u}_b^h\}\,\middle|\,\mathbf{u}_0^h\big|_K\in[P_i(K)]^d,\mathbf{u}_b^h\big|_{\partial K}\in[P_{i-1}(K)]^d\text{ for all }K\in K_h\right\}$$

and denote $\mathbf{V}_{1h}^0=\left\{\mathbf{u}_h=\{\mathbf{u}_0^h,\mathbf{u}_b^h\}\,\middle|\,\mathbf{u}_h\in\mathbf{V}_{1h},\mathbf{u}_b^h\big|_{\partial K\cap\partial\Omega}=\mathbf{0}\right\}$; for the magnetic field variable,

$$\mathbf{V}_{2h}=\left\{\mathbf{B}_h=(\mathbf{B}_0^h,\mathbf{B}_b^h)\,\middle|\,\mathbf{B}_0^h\big|_K\in[P_i(K)]^d,\mathbf{B}_b^h\big|_{\partial K}\in[P_{i-1}(K)]^d\text{ for all }K\in K_h\right\}$$

and denote $\mathbf{V}_{2h}^0=\left\{\mathbf{B}_h=\{\mathbf{B}_0^h,\mathbf{B}_b^h\}\,\middle|\,\mathbf{B}_h\in\mathbf{V}_{2h},\ \mathbf{B}_b^h\cdot\mathbf{n}\big|_{\partial K\cap\partial\Omega}=\mathbf{0}\right\}$.

Moreover, we denote $\mathbf{y}_h=\{\mathbf{y}_0^h,\mathbf{y}_b^h\}$ by $\mathbf{y}_0^h=\{\mathbf{u}_0^h,\mathbf{B}_0^h\},\mathbf{y}_b^h=\{\mathbf{u}_b^h,\mathbf{B}_b^h\}$.

In the further, we shall drop the subscript r and K to simplify the notations for the discrete weak gradient, divergence and rotation operators.

To investigate the approximation properties of the discrete weak Galerkin finite element spaces $\mathbf{V}_{1h}$ and $\mathbf{V}_{2h}$, we use three projection operators: $Q_h\mathbf{v}=\{Q_0\mathbf{v}, Q_b\mathbf{v}\}$ is L^2 projection operator from $\mathbf{W}_2^1$ onto $\mathbf{V}_{1h}$ or $\mathbf{V}_{2h}$, $\mathbf{R}_h\mathbf{v}$ is L^2 projection operator onto $[P_i(K)]^{d\times d}$, $R_h\mathbf{v}$ is L^2 projection operator onto $P_i(K)$ and $\Pi_h\mathbf{v}$ is L^2 projection operator from $\mathbf{H}(div,\Omega)$ onto $\mathbf{H}(div,\Omega)$, $\Pi_h\mathbf{v}\in[P_i(K)]^d$ satisfies

$$(div\mathbf{v}, v_0)_K=(div\Pi_h\mathbf{v}, v_0)_K \qquad \forall v_0\in P_i(K) \tag{3.1}$$

For $\mathbf{y}=\{\mathbf{u},\mathbf{B}\}\in\mathbf{V}=\mathbf{V}_1\times\mathbf{V}_2$, we set $Q_h\mathbf{y}=\{Q_h\mathbf{u}, Q_h\mathbf{B}\}$, $\mathbf{R}_h\mathbf{y}=\{\mathbf{R}_h\mathbf{u},\mathbf{R}_h\mathbf{B}\}$, $R_h\mathbf{y}=\{R_h\mathbf{u}, R_h\mathbf{B}\}$ and $\Pi_h\mathbf{y}=\{\Pi_h\mathbf{u},\Pi_h\mathbf{B}\}$.

The L^2 projection operators Q_h, $\mathbf{R}_h$ and R_h satisfy the following properties.

Lemma 3.1. For any $\mathbf{y}=\{\mathbf{u},\mathbf{B}\}\in\mathbf{V}=\mathbf{V}_1\times\mathbf{V}_2$, The L^2 projection operators Q_h, $\mathbf{R}_h$ and R_h satisfy

$$\nabla_w Q_h\mathbf{y}=\mathbf{R}_h(\nabla\mathbf{y}),\quad div_w Q_h\mathbf{y}=R_h(div\mathbf{y}),\ rot_w Q_h\mathbf{y}=\mathbf{R}_h(rot\mathbf{y}) \tag{3.2}$$

$$\left\|\mathbf{y}-Q_h^0\mathbf{y}\right\|_{0,K}\le Ch_K^s\|\mathbf{y}\|_{s,K} \qquad 0\le s\le i,\ K\in K_h, \tag{3.3}$$

$$\left\|\nabla_w Q_h\mathbf{y}-\nabla\mathbf{y}\right\|_{0,K}\le Ch_K^s\|\mathbf{y}\|_{s+1,K} \quad 0\le s\le i+1,\ K\in K_h \tag{3.4}$$

Proof. From Definition 1, L^2 projection operator $Q_h\mathbf{y}$ and Green's formula, we have

$$\int_K \nabla_w Q_h\mathbf{y}\cdot\mathbf{q}dx=-\int_K(Q_0\mathbf{y})\nabla\cdot\mathbf{q}dx+\int_{\partial K}(Q_b\mathbf{y})\mathbf{q}\cdot\mathbf{n}ds,\quad \forall\mathbf{q}\in[P_i(K)]^{d\times d} \tag{3.5}$$

Since Q_0 and Q_b are L^2-projection operator, then the right-hand side of (3.5) is given by

$$-\int_K(Q_0\mathbf{y})\nabla\cdot\mathbf{q}dx+\int_{\partial K}(Q_b\mathbf{y})\mathbf{q}\cdot\mathbf{n}ds=-\int_K\mathbf{y}\nabla\cdot\mathbf{q}dx+\int_{\partial K}\mathbf{y}\mathbf{q}\cdot\mathbf{n}ds=\int_K(\nabla\mathbf{y})\cdot\mathbf{q}dx=\int_K\mathbf{R}_h(\nabla\mathbf{y})\cdot\mathbf{q}dx$$

This shows that $\nabla_w Q_h\mathbf{y}=\mathbf{R}_h(\nabla\mathbf{y})$ holds.

Similarly, from Definition 2, we can derive $div_w Q_h\mathbf{y}=R_h(div\mathbf{y})$ and $rot_w Q_h\mathbf{y}=R_h(rot\mathbf{y})$.

Also, it implies approximation property (3.3) and estimate (3.4).

Lemma 3.2. For any $q\in\mathbf{H}(div,\Omega)\times\mathbf{H}(div,\Omega)$,

$$\sum_{K\in K_h}(-\nabla\cdot q,\mathbf{y}_0)_K=\sum_{K\in K_h}(\Pi_h q,\ \nabla_w\mathbf{y})_K,\qquad \forall\mathbf{y}=(\mathbf{y}_0,\mathbf{y}_b)\in\mathbf{V}_{1h}^0\times\mathbf{V}_{2h}^0 \tag{3.6}$$

Proof. From L^2 projection Π_h and Definition 1,

$$\sum_{K\in K_h}(-\nabla\cdot q,\mathbf{y}_0)_K=\sum_{K\in K_h}(-\nabla\cdot\Pi_h q,\ \mathbf{y}_0)_K=$$

$$=\sum_{K\in K_h}(\Pi_h q,\ \nabla_w\mathbf{y})_T-\sum_{K\in K_h}(\mathbf{y}_b,\Pi_h q\cdot n)_{\partial K}=\sum_{K\in K_h}(\Pi_h q,\ \nabla_w\mathbf{y})_K$$

The weak Galerkin finite element method is to replace the classical gradient and rotation operators by the weak gradient operator ∇_w and weak rotation operator rot_w and to use the discrete weak finite element space $\mathbf{V}_{1h}^0\times\mathbf{V}_{2h}^0$.

First, we introduce the semi-discrete WG finite element method.

The semi-discrete WG finite element method of (2.8) – (2.9) is to find $\mathbf{y}_h(x,t)\in\mathbf{V}_{1h}^0\times\mathbf{V}_{2h}^0$, satisfying $\mathbf{u}_b^h=\mathbf{0}$ and $\mathbf{B}_b^h\cdot\mathbf{n}=0$ on $\partial\Omega,\ t\in[0,T]$ for $Q_b\mathbf{y}=\mathbf{y}_b^h=\{\mathbf{u}_b^h,\mathbf{B}_b^h\}$, $\mathbf{y}_h(x,0)=Q_h\mathbf{y}_0$ in Ω and the following equation

$$(\dot{\mathbf{y}}_h, \mathbf{z}_0) + A(\mathbf{y}_h, \mathbf{z}) + (G[\mathbf{y}_h], \mathbf{z}) = (F(\mathbf{J}_c), \mathbf{z}_0), \ \forall \mathbf{z} \in \mathbf{V}_{1h}^0 \times \mathbf{V}_{2h}^0, \ t \in [0, T] \tag{3.7}$$

where $(A\mathbf{y}, \mathbf{z})$ and $(G[\mathbf{y}], \mathbf{z})$ are respectively the weak bilinear and trilinear form defined by

$$A(\mathbf{y}, \mathbf{z}) = a_1(\mathbf{u}, \mathbf{v}) + a_2(\mathbf{B}, \ \mathbf{w})$$

$$a_1(\mathbf{u}, \mathbf{v}) = \gamma(\nabla_w \mathbf{u}, \nabla_w \mathbf{v}) = \gamma(rot_w \mathbf{u}, \ rot_w \mathbf{v})$$

$$a_2(\mathbf{B}, \mathbf{w}) = \gamma_\mu(rot_w \mathbf{B}, \ rot_w \mathbf{w})$$

$$(G_w[\mathbf{y}], \mathbf{z}) = (rot_w \mathbf{u} \times \mathbf{u} - rot_w \mathbf{B} \times \mathbf{B}, \mathbf{v}) - (\mathbf{u} \times \mathbf{B}, rot_w \mathbf{w})$$

and $(F(\mathbf{J}_c), \mathbf{z}_0)$ is defined by $(F(\mathbf{J}_c), \mathbf{z}_0) = (\mathbf{J}_c, rot_w \mathbf{w}_0)$.

For the equation (3.7), we can get the following splitting equation, respectively:

$$\begin{cases} \left(\dfrac{\partial \mathbf{u}_h}{\partial t}, \mathbf{v}_0\right) + \gamma \cdot (\nabla_w \mathbf{u}_h, \nabla_w \mathbf{v}) + (rot_w \mathbf{u}_h \times \mathbf{u}_h, \mathbf{v}) - (rot_w \mathbf{B}_h \times \mathbf{B}_h, \mathbf{v}) = \mathbf{0} \\ \left(\dfrac{\partial \mathbf{B}_h}{\partial t}, \mathbf{w}_0\right) + \gamma_\mu (rot_w \mathbf{B}_h, rot_w \mathbf{w}) - (\mathbf{u}_h \times \mathbf{B}_h, rot_w \mathbf{w}) = (\mathbf{J}_c, rot_w \mathbf{w}_0), \end{cases} \tag{3.8}$$

where

$$(\nabla_w \mathbf{u}, \nabla_w \mathbf{v}) = \sum_{K \in K_h} (\nabla_w \mathbf{u}, \nabla_w \mathbf{v})_K, (rot_w \mathbf{B}, rot_w \mathbf{w}) = \sum_{K \in K_h} (rot_w \mathbf{B}, \ rot_w \mathbf{w})_K, (rot_w \mathbf{u}, \mathbf{v}) = \sum_{K \in K_h} (rot_w \mathbf{u}, \mathbf{v})_K.$$

Next, we turn our attention to the full discrete WG finite element method.

Let τ be the time step size and $t_n = n\tau$, where n is a nonnegative integer.

We denote by $\mathbf{y}_h(t_n) = \mathbf{y}_h^n$ the approximation of $\mathbf{y}(t_n)$.

Writing the full discrete WG finite element scheme for equation (2.8) – (2.9),

$$\begin{aligned} &(\partial_t \mathbf{y}_h^n, \mathbf{z}_0) + A(\mathbf{y}_h^n, \mathbf{z}) + (G_w[\mathbf{y}_h^n], \mathbf{z}) = (F(\mathbf{J}_c), \mathbf{z}_0), \ \forall \mathbf{z} \in \mathbf{V}_{1h}^0 \times \mathbf{V}_{2h}^0, \\ &\mathbf{y}_h^0 = Q_h \mathbf{y}_0 \end{aligned} \tag{3.9}$$

where $\partial_t \mathbf{y}_h^k = \dfrac{\mathbf{y}_h^{k+1} - \mathbf{y}_h^k}{\tau}$ or $\partial_t \mathbf{y}_h^k = \dfrac{\mathbf{y}_h^k - \mathbf{y}_h^{k-1}}{\tau}$.

For the equation (3.9), we can get the following splitting equation, respectively:

$$\begin{aligned} &\begin{cases} \left(\dfrac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau}, \mathbf{v}_0\right) + \gamma \cdot (\nabla_w \mathbf{u}_h^n, \nabla_w \mathbf{v}) + (rot_w \mathbf{u}_h^n \times \mathbf{u}_h^n, \mathbf{v}) - (rot_w \mathbf{B}_h^n \times \mathbf{B}_h^n, \mathbf{v}) = \mathbf{0} \\ \left(\dfrac{\mathbf{B}_h^n - \mathbf{B}_h^{n-1}}{\tau}, \mathbf{w}_0\right) + \gamma_\mu (rot_w \mathbf{B}_h^n, rot_w \mathbf{w}) - (\mathbf{u}_h^n \times \mathbf{B}_h^n, rot_w \mathbf{w}) = (\mathbf{J}_c, rot_w \mathbf{w}_0) \end{cases} \\ &\mathbf{u}_h^0 = Q_h \mathbf{u}_0, \ \mathbf{B}_h^0 = Q_h \mathbf{B}_0 \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} &\begin{cases} \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\tau}, \mathbf{v}_0\right) + \gamma \cdot (\nabla_w \mathbf{u}_h^n, \nabla_w \mathbf{v}) + (rot_w \mathbf{u}_h^n \times \mathbf{u}_h^n, \mathbf{v}) - (rot_w \mathbf{B}_h^n \times \mathbf{B}_h^n, \mathbf{v}) = \mathbf{0} \\ \left(\dfrac{\mathbf{B}_h^{n+1} - \mathbf{B}_h^n}{\tau}, \mathbf{w}_0\right) + \gamma_\mu (rot_w \mathbf{B}_h^n, rot_w \mathbf{w}) - (\mathbf{u}_h^n \times \mathbf{B}_h^n, rot_w \mathbf{w}) = (\mathbf{J}_c, rot_w \mathbf{w}_0) \end{cases} \\ &\mathbf{u}_h^0 = Q_h \mathbf{u}_0, \ \mathbf{B}_h^0 = Q_h \mathbf{B}_0 \end{aligned} \tag{3.11}$$

This approximate solutions $\mathbf{u}_h^k$ and $\mathbf{B}_h^k$ are called the approximate solutions of the full discrete WG finite element method for the equations (2.8) – (2.9).

Remark 1. In this article, we haven`t indicated for the hydrodynamic pressure $p = p(x, t)$, because we can find easily the hydrodynamic pressure $p = p(x, t)$ from the equation (2.1) if we have found out the approximate solutions $\mathbf{u}_h^k$ and $\mathbf{B}_h^k$ in the equation (3.8) or (3.10).

4. Error analysis of approximate solution

In this section, we derive some error estimates for semi-discrete and full discrete WG finite element methods.

By means of projections Q_h and $\mathbf{R}_h$, we can derive the following approximation property.

Lemma 4.1. For $\mathbf{u}, \mathbf{B} \in \mathbf{H}^{1+s}(\Omega)$ with $s > 0$, we have respectively

$$\|\Pi_h(\nabla\mathbf{u}) - \nabla_w(Q_h\mathbf{u})\| \le ch^s\|\mathbf{u}\|_{1+s},\ \|\Pi_h(rot\mathbf{B}) - rot_w(Q_h\mathbf{B})\| \le ch^s\|\mathbf{B}\|_{1+s} \tag{4.1}$$

$$\|\nabla\mathbf{u} - \nabla_w(Q_h\mathbf{u})\| \le ch^s\|\mathbf{u}\|_{1+s},\ \|rot\mathbf{B} - rot_w(Q_h\mathbf{B})\| \le ch^s\|\mathbf{B}\|_{1+s} \tag{4.2}$$

Proof. Since from (3.2) we have $\nabla_w Q_h\mathbf{u} = \mathbf{R}_h(\nabla\mathbf{u})$ and $\nabla_w Q_h\mathbf{B} = \mathbf{R}_h(\nabla\mathbf{B})$, then

$$\|\Pi_h(\nabla\mathbf{u}) - \nabla_w(Q_h\mathbf{u})\| = \|\Pi_h(\nabla\mathbf{u}) - \mathbf{R}_h(\nabla\mathbf{u})\|$$

Using the triangle inequality and the definition of Π_h, we have

$$\|\Pi_h(\nabla\mathbf{u}) - \mathbf{R}_h(\nabla\mathbf{u})\| \le \|\Pi_h(\nabla\mathbf{u}) - \nabla\mathbf{u}\| + \|\nabla\mathbf{u} - \mathbf{R}_h(\nabla\mathbf{u})\| \le$$

$$\le c_1h^{1+s}\|\nabla\mathbf{u}\|_{1+s} + c_2h^{1+s}\|\nabla\mathbf{u}\|_{1+s} \le ch^s\|\mathbf{u}\|_{1+s}$$

Similarly, we can derive $\|\Pi_h(rot\mathbf{B}) - rot_w(Q_h\mathbf{B})\| \le ch^s\|\mathbf{B}\|_{1+s}$.

The estimate (4.2) can be derived in a similar way. This completes a proof of the lemma.

Next, we shall prove the following estimate for the error of the semi-discrete solution.

Next, we introduce following lemma. ([26])

Lemma 4.2. Assume that $\mathbf{y} = \{\mathbf{u}, \mathbf{B}\} = \{\mathbf{y}_0, \mathbf{y}_b\} \in \mathbf{V}_{1h}^0 \times \mathbf{V}_{2h}^0$ $(\mathbf{y}_0 = \{\mathbf{u}_0, \mathbf{B}_0\}, \mathbf{y}_b = \{\mathbf{u}_b, \mathbf{B}_b\})$, then there exist a constant C such that

$$\|\mathbf{y}\| \le C\|rot_w\mathbf{y}\| \tag{4.3}$$

where $\nabla_w\mathbf{y} \in \mathbf{V}_{1h} \times \mathbf{V}_{2h}$.

Theorem 4.3. Let $\mathbf{y} \in \mathbf{H}^{1+s}(\Omega)$ and $\mathbf{y}_h$ be the solutions of equations (2.8)-(2.9) and (3.7), respectively. Then, there exists a constant $\tilde{c}$ such that

$$\|\mathbf{y}_h(t) - Q_h\mathbf{y}(t)\|^2 \le \tilde{c}h^{2s}\int_0^t\|\mathbf{y}\|_{1+s}\,dt \tag{4.4}$$

where $\mathbf{y}_h(t) = \{\mathbf{u}_h(t), \mathbf{B}_h(t)\}$, $Q_h\mathbf{y}(t) = \{Q_h\mathbf{u}, Q_h\mathbf{B}\}$.

Proof. Let $\mathbf{z} \in \{\mathbf{z}_0, \mathbf{z}_b\} \in \mathbf{V}_{1h}^0 \times \mathbf{V}_{2h}^0$ be the testing function.

For $\mathbf{y} \in H^{1+s}(\Omega)$, we know that $rot_w(Q_h\mathbf{y}) = \mathbf{R}_h(rot\mathbf{y}), (Q_h\dot{\mathbf{y}}, \mathbf{z}_0) = (\dot{\mathbf{y}}, \mathbf{z}_0)$ and $(Q_h\mathbf{y}, \mathbf{z}_0) = (\mathbf{y}, \mathbf{z}_0)$.

For $\mathbf{y} \in H^{1+s}(\Omega)$, we obtain

$$(F(\mathbf{J}_c),\ \mathbf{z}_0) = (\dot{\mathbf{y}},\ \mathbf{z}_0) + \sum_{K\in K_h} A(\mathbf{y}, \mathbf{z})_K + (G_w(\mathbf{y}, \mathbf{y}), \mathbf{z}) =$$

$$= (\dot{\mathbf{y}}, \mathbf{z}_0) + (\Pi_h(rot\mathbf{y}), rot_w\mathbf{z}) + (G_w(\mathbf{y}, \mathbf{y}), \mathbf{z}) =$$
$$= (Q_h\dot{\mathbf{y}}, \mathbf{z}_0) + (rot_w(Q_h\mathbf{y}),\ rot_w\mathbf{z}) + (\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y}), \nabla_w\mathbf{z}) + (G_w(Q_h\mathbf{y}, Q_h\mathbf{y}), \mathbf{z}).$$

Also, as the solution $\mathbf{y}_h$ of equation (3.7), we have

$$(F(\mathbf{J}_c), \mathbf{z}_0) = (\dot{\mathbf{y}}_h, \mathbf{z}_0) + (rot_w\mathbf{y}_h,\ rot_w\mathbf{z}) + (G_w(\mathbf{y}_h, \mathbf{y}_h), \mathbf{z})$$

Combining the above two equations, we get

$$\begin{aligned}(\dot{\mathbf{y}}_h - Q_h\dot{\mathbf{y}},\ \mathbf{z}_0) + (rot_w(\mathbf{y}_h - Q_h\mathbf{y}), rot_w\mathbf{z}) =\\ = (\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y}),\ \nabla_w\mathbf{z}) + (G_w(Q_h\mathbf{y}, Q_h\mathbf{y}), \mathbf{z}) - (G_w(\mathbf{y}_h, \mathbf{y}_h), \mathbf{z})\end{aligned} \tag{4.5}$$

Now, Denote by $\mathbf{e}_h := \mathbf{y}_h - Q_h\mathbf{y}$ the difference between the weak Galekin approximation and the L^2 projection of the exact solution $\mathbf{y}$.

Substituting $\mathbf{e}_h$ for $\mathbf{z}$ in (4.4) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(\dot{\mathbf{e}}_h, \mathbf{e}_{h0}) + (rot_w\mathbf{e}_h, rot_w\mathbf{e}_h) = (\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y}), rot_w\mathbf{e}_h) + (G_w(Q_h\mathbf{y}, Q_h\mathbf{y}), \mathbf{e}_h) -\\
- (G_w(\mathbf{y}_h, Q_h\mathbf{y}), \mathbf{e}_h) + (G_w(\mathbf{y}_h, Q_h\mathbf{y}), \mathbf{e}_h) - (G_w(\mathbf{y}_h, \mathbf{y}_h), \mathbf{e}_h) =\\
= (\Pi_h(rot\nabla\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y}), rot_w\mathbf{e}_h) - (G_w(\mathbf{e}_h, Q_h\mathbf{y}), \mathbf{e}_h) \le\\
\le \|\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y})\| \cdot \|rot_w\mathbf{e}_h\| + c|\mathbf{e}_h|^{\frac{3}{4}} \cdot \|\mathbf{e}_h\|^{\frac{1}{4}} \cdot |Q_h\mathbf{y}| \cdot |\mathbf{e}_h|^{\frac{3}{4}} \cdot \|\mathbf{e}_h\|^{\frac{1}{4}} \le\\
\le \|\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y})\| \cdot \|rot_w\mathbf{e}_h\| + c|\mathbf{e}_h|^{\frac{3}{2}} \cdot \|\mathbf{e}_h\|^{\frac{1}{2}} \cdot |Q_h\mathbf{y}| \le\\
\le \|\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y})\| \cdot \|rot_w\mathbf{e}_h\| + c\|rot_w\mathbf{e}_h\|^{\frac{3}{2}} \cdot \|\mathbf{e}_h\|^{\frac{1}{2}} \cdot \|\mathbf{y}\| \le\\
\le \|\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y})\|^2 + \frac{1}{4}\|rot_w\mathbf{e}_h\|^2 + \frac{3}{4}\|rot_w\mathbf{e}_h\|^2 + \frac{1}{4}c_1\|\mathbf{e}\|^2 =\\
= \|\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y})\|^2 + \|rot_w\mathbf{e}_h\|^2 + c_2\|\mathbf{e}_h\|^2\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{e}_h\|^2 + \|rot_w\mathbf{e}_h\|^2 \le \|\Pi_h(rot\mathbf{y}) - \mathbf{R}_h(rot\mathbf{y})\|^2 + \|rot_w\mathbf{e}_h\|^2 + c_2\|\mathbf{e}_h\|^2.$$

We obtain by Lemma 4.1

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{e}_h\|^2 \le c_2\|\mathbf{e}_h\|^2 + c'h^{2s}\|\mathbf{y}\|_{s+1}^2. \tag{4.6}$$

Thus, integrating with respect to t and from $e(\cdot, 0) = 0$, we arrive at

$$\|\mathbf{e}_h\|^2 \le 2c'h^{2s}\int_0^t \|\mathbf{y}\|_{s+1}^2 dt + 2c_2\int_0^t \|\mathbf{e}_h\|^2 dt \tag{4.7}$$

By Gronwall's inequality

$$\|\mathbf{e}_h\|^2 \le c\mathbf{e}^{ct}\, 2c'h^{2s}\int_0^t \|\mathbf{y}\|_{s+1}^2 dt \tag{4.8}$$

That is,

$$\|\mathbf{e}_h\|^2 \le \tilde{c} \cdot h^{2s}\int_0^t \|\mathbf{y}\|_{s+1}^2 dt$$

The proof is completed.

Now, we shall derive an error estimate for the full discrete WG approximation.

Theorem 4.4. Let $\mathbf{y} \in \mathbf{H}^{1+s}(\Omega)$ and $\mathbf{y}_h^n$ be the solutions of equations (2.8)-(2.9) and (3.10), respectively. Denote by $\mathbf{e}^n := \mathbf{y}_h^n - Q_h \mathbf{y}(t_n)$ the difference between the full discrete WG approximation and the L^2 projection of the exact solution $\mathbf{y}$.

Then, there exists a constant C such that

$$\left\|\mathbf{e}^n\right\|^2 + \tau \sum_{i=1}^{n} \left\| rot_w \mathbf{e}^i \right\|^2 \le \left\|\mathbf{e}^0\right\|^2 + C\left(h^{2r} \cdot \tau \|\mathbf{y}\|_{1+r}^2 + \tau^2 \int_0^{t_n} \left\| \frac{\partial^2 \mathbf{y}(s)}{\partial t^2} \right\|^2 ds \right) \qquad (n > 0) \qquad (4.9)$$

where $\mathbf{y}_h^n = \mathbf{y}_h(t_n) = \{\mathbf{u}_h(t_n), \mathbf{B}_h(t_n)\}$, $Q_h \mathbf{y}(t_n) = \{Q_h \mathbf{u}, Q_h \mathbf{B}\}$.

Proof. From the equations (2.8) and (3.9),

$$(\dot{\mathbf{y}}, \mathbf{z}_0) + A(\mathbf{y}, \mathbf{z}) + G(\mathbf{y}, \mathbf{y}, \mathbf{z}) = \left(F(\mathbf{J}_c), z_0\right) \qquad (4.10)$$

$$\left(\frac{\mathbf{y}_h^n - \mathbf{y}_h^{n-1}}{\tau}, \mathbf{z}_0 \right) + A(\mathbf{y}_h^n, \mathbf{z}) + G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{z}\right) = \left(F(\mathbf{J}_c), z_0\right) \qquad (4.11)$$

Subtracting the equation (4.10) from the equation (4.11)

$$\left(\frac{\mathbf{y}_h^n - \mathbf{y}_h^{n-1}}{\tau} - \dot{\mathbf{y}}(t_n), \mathbf{z}_0 \right) + A(\mathbf{y}_h^n, \mathbf{z}) - A(\mathbf{y}(t_n), \mathbf{z}) = G\left(\mathbf{y}(t_n), \mathbf{y}(t_n), \mathbf{z}\right) - G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{z}\right) \quad (4.12)$$

Writing the equation (4.12) similarly with the equation (4.5),

$$\left(\frac{\mathbf{y}_h^n - \mathbf{y}_h^{n-1}}{\tau} - \frac{Q_h \mathbf{y}(t_n) - Q_h \mathbf{y}(t_{n-1})}{\tau}, \mathbf{z}_0 \right) + \left(\frac{Q_h \mathbf{y}(t_n) - Q_h \mathbf{y}(t_{n-1})}{\tau} - \dot{\mathbf{y}}(t_n), \mathbf{z}_0 \right) + A(\mathbf{y}_h^n, \mathbf{z}) - A(\mathbf{y}(t_n), \mathbf{z}) =$$
$$= \left(\Pi_h(rot\mathbf{y}(t_n)) - \mathbf{R}_h(rot\mathbf{y}(t_n)), rot_w \mathbf{z}\right) + G\left(\mathbf{y}(t_n), \mathbf{y}(t_n), \mathbf{z}\right) - G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{z}\right)$$

Rewriting the above equation,

$$\left(\frac{\mathbf{y}_h^n - \mathbf{y}_h^{n-1}}{\tau} - \frac{Q_h \mathbf{y}(t_n) - Q_h \mathbf{y}(t_{n-1})}{\tau}, \mathbf{z}_0 \right) + \left(\frac{Q_h \mathbf{y}(t_n) - Q_h \mathbf{y}(t_{n-1})}{\tau} - \dot{\mathbf{y}}_h(t_n), \mathbf{z}_0 \right) + A(\mathbf{y}_h^n, \mathbf{z}) -$$
$$- A(Q_h \mathbf{y}(t_n), \mathbf{z}) = \left(\Pi_h(rot\mathbf{y}(t_n)) - \mathbf{R}_h(rot\mathbf{y}(t_n)), rot_w \mathbf{z}\right) + G\left(\mathbf{y}(t_n), y(t_n), \mathbf{z}\right) - G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{z}\right)$$

Taking account of $\mathbf{e}^n = \mathbf{y}_h^n - Q_h \mathbf{y}(t_n)$, we obtain

$$\left(\frac{\mathbf{e}^n - \mathbf{e}^{n-1}}{\tau}, \mathbf{z}_0 \right) + A(\mathbf{e}^n, \mathbf{z}) = \left(\dot{\mathbf{y}}_h(t_n) - \frac{\mathbf{y}(t_n) - \mathbf{y}(t_{n-1})}{\tau}, \mathbf{z}_0 \right) + \left(\Pi_h(rot\mathbf{y}(t_n)) - \mathbf{R}_h(rot\mathbf{y}(t_n)), rot_w \mathbf{z}\right) +$$
$$+ G\left(\mathbf{y}(t_n), \mathbf{y}(t_n), \mathbf{z}\right) - G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{z}\right)$$

Also, let $\mathbf{w}_1^n = \dot{\mathbf{y}}(t_n) - \dfrac{\mathbf{y}(t_n) - \mathbf{y}(t_{n-1})}{\tau}$, $\mathbf{w}_2^n = \Pi_h\left(rot\mathbf{y}(t_n)\right) - \mathbf{R}_h\left(rot\mathbf{y}(t_n)\right)$ and choosing the $z = e^n$,

$$\left(\frac{\mathbf{e}^n - \mathbf{e}^{n-1}}{\tau}, \mathbf{e}^n \right) + A(\mathbf{e}^n, \mathbf{e}^n) = (\mathbf{w}_1^n, \mathbf{e}^n) + (\mathbf{w}_2^n, rot_w \mathbf{e}^n) + G\left(\mathbf{y}(t_n), \mathbf{y}(t_n), \mathbf{e}^n\right) - G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{e}^n\right)$$

Therefore, we have

$$\left\|\mathbf{e}^n\right\|^2 - (\mathbf{e}^{n-1}, \mathbf{e}^n) + \tau \left\| rot_w \mathbf{e}^n \right\|^2 \le \tau \left\|\mathbf{w}_1^n\right\| \cdot \left\|\mathbf{e}^n\right\| + \tau \left\|\mathbf{w}_2^n\right\| \cdot \left\|\nabla_w \mathbf{e}^n\right\| + \tau G\left(Q_h \mathbf{y}(t_n), Q_h \mathbf{y}(t_n), \mathbf{e}^n\right) - G_w\left(\mathbf{y}_h^n, \mathbf{y}_h^n, \mathbf{e}^n\right) =$$
$$= \tau \left\|\mathbf{w}_1^n\right\| \cdot \left\|\mathbf{e}^n\right\| + \tau \left\|\mathbf{w}_2^n\right\| \cdot \left\|\nabla_w \mathbf{e}^n\right\| + \tau G\left(\mathbf{e}^n, Q_h \mathbf{y}(t_n), \mathbf{e}^n\right) \le$$
$$\le \tau \left\|\mathbf{w}_1^n\right\| \cdot \left\|\mathbf{e}^n\right\| + \tau \left\|\mathbf{w}_2^n\right\| \cdot \left\|\nabla_w \mathbf{e}^n\right\| + c\,\tau \left|\mathbf{e}^n\right|^{\frac{3}{4}} \cdot \left\|\mathbf{e}^n\right\|^{\frac{1}{4}} \cdot \left|Q_h \mathbf{y}(t_n)\right| \cdot \left|\mathbf{e}^n\right|^{\frac{3}{4}} \cdot \left\|\mathbf{e}^n\right\|^{\frac{1}{4}}.$$

Then,

$$\|\mathbf{e}^n\|^2 + \tau\|rot_w \mathbf{e}^n\|^2 \le (\mathbf{e}^{n-1}, \mathbf{e}^n) + \tau\|\mathbf{w}_1^n\| \cdot \|\mathbf{e}^n\| + \tau\|\mathbf{w}_2^n\| \cdot \|\nabla_w \mathbf{e}^n\| + c\tau|\mathbf{e}^n|^{\frac{3}{2}} \cdot \|\mathbf{e}^n\|^{\frac{1}{2}} \cdot |\mathbf{y}(t_n)|$$

By the boundedness of $\mathbf{y}$ and the Poincare` inequality, it follows

$$\|\mathbf{e}^n\|^2 + \tau\|rot_w \mathbf{e}^n\|^2 \le \frac{1}{2}\|\mathbf{e}^{n-1}\|^2 + \frac{1}{2}\|\mathbf{e}^n\|^2 + \tau\left(\|\mathbf{w}_1^n\| + \|\mathbf{w}_2^n\| + c_1\right) \cdot \|rot_w \mathbf{e}^n\|.$$

Consequently,

$$\frac{1}{2}\|\mathbf{e}^n\|^2 + \tau\|rot_w \mathbf{e}^n\|^2 \le \frac{1}{2}\|\mathbf{e}^{n-1}\|^2 + \frac{\tau}{2}\left(\|\mathbf{w}_1^n\| + \|\mathbf{w}_2^n\| + c_1\right)^2 + \frac{\tau}{2}\|rot_w \mathbf{e}^n\|^2.$$

That is,

$$\frac{1}{2}\|\mathbf{e}^n\|^2 + \frac{\tau}{2}\|rot_w \mathbf{e}^n\|^2 \le \frac{1}{2}\|\mathbf{e}^{n-1}\|^2 + \frac{c_2\tau}{2}\left(\|\mathbf{w}_1^n\|^2 + \|\mathbf{w}_2^n\|^2 + c_1{}^2\right)$$

By repeated application,

$$\|\mathbf{e}^n\|^2 + \frac{\tau}{2}\|rot_w \mathbf{e}^n\|^2 \le \|\mathbf{e}^0\|^2 + c_2\tau\left(\sum_{i=1}^{n}\|\mathbf{w}_1^i\|^2 + \sum_{i=1}^{n}\|\mathbf{w}_2^i\|^2 + c_1{}^2\right) \qquad (4.13)$$

By Lemma 4.1 and $\mathbf{R}_h(rot\mathbf{y}) = rot_w(Q_h\mathbf{y})$,

$$\|\mathbf{w}_2^i\| \le c_3 h^s \|\mathbf{y}(t_i)\|_{s+1}.$$

From $\tau\, \mathbf{w}_1^i = \tau\frac{\partial \mathbf{y}(t_i)}{\partial t} - (\mathbf{y}(t_i) - \mathbf{y}(t_{i-1})) = \int_{t_{i-1}}^{t_i}(t - t_{i-1})\frac{\partial^2 \mathbf{y}(t)}{\partial t^2}dt$,

$$\mathbf{w}_1^i = \frac{1}{\tau}\int_{t_{i-1}}^{t_i}(t - t_{i-1})\frac{\partial^2 \mathbf{y}(t)}{\partial t^2}dt \qquad (4.14)$$

Thus,

$$\|\mathbf{w}_1^i\|^2 = \int_\Omega\left(\frac{1}{\tau}\int_{t_{i-1}}^{t_i}(t - t_{i-1})\frac{\partial^2 \mathbf{y}(t)}{\partial t^2}ds\right)^2 d\Omega \le \frac{1}{\tau^2}\int_\Omega\left[\int_{t_{i-1}}^{t_i}(t - t_{i-1})^2 dt \cdot \int_{t_{i-1}}^{t_i}\left(\frac{\partial^2 \mathbf{y}(t)}{\partial t^2}\right)^2 dt\right]d\Omega \le$$

$$\le c_4\tau \cdot \int_{t_{i-1}}^{t_i}\left\|\frac{\partial^2 \mathbf{y}(t)}{\partial t^2}\right\|^2 dt \qquad (4.15)$$

Then by substituting (4.15) into (4.13), we have

$$\|\mathbf{e}^n\|^2 + \tau\sum_{i=1}^{n}\|rot_w \mathbf{e}^i\|^2 \le \|\mathbf{e}^0\|^2 + C\left(h^{2r} \cdot \tau\|\mathbf{y}\|_{1+r}^2 + \tau^2\int_0^{t_n}\left\|\frac{\partial^2 \mathbf{y}(t)}{\partial t^2}\right\|^2 ds\right) \qquad (n > 0)$$

The proof is completed.

5. Numerical experiment

In this section, we present the results of numerical experiment. We carry out benchmark test in 2D.

We consider a simple problem for the incompressible viscous MHD equations with known analytical solution in 2D.

We choose $\Omega = [0, 1] \times [0, 1]$ and time interval $\mathbf{I} = [0, 1]$ with computational domain.

Also we assume that problem (2.1)-(2.5) has an analytic solution, which is given by $\mathbf{u}=(e^{-t}\sin 2\pi x\cos 2\pi y,\ -e^{-t}\cos 2\pi x\sin 2\pi y)^T$, $\mathbf{B}=(e^{-t}\sin 2\pi y,\, e^{-t}\sin 2\pi x)^T$,

By simple computation, we use the parameters $\gamma=\frac{1}{8\pi^2}$, $\gamma_\mu=\frac{1}{4\pi^2}$, $a=1$ in the tests.

We choose uniform triangular mesh and let $h=1/N\ (N=8, 16, 32, 64, 128)$ be mesh sizes for triangular meshes.

Let $\mathbf{u}$ and $\mathbf{u}_h$ be the exact velocity and the WG finite element approximation, $\mathbf{B}$ and $\mathbf{B}_h$ be the exact magnetic field and the WG finite element approximation.

In the test, $\tau=h$ and $\tau=h^2$ are used to check the order of convergence with respect to time step size τ and mesh size h.

The results are shown in Tables ***I*** and ***II***.

Table *I*

Result of WG finite element method with $\gamma=1/8\pi^2$, $\gamma_\mu=1/4\pi^2$, $a=1$ and $\tau=h$.

$1/h$	$\|\nabla(\mathbf{u}-\mathbf{u}_h^n)\|$	$\|\mathbf{u}-\mathbf{u}_h^n\|$	$\|\nabla(\mathbf{B}-\mathbf{B}_h^n)\|$	$\|\mathbf{B}-\mathbf{B}_h^n\|$
8	1.054e-2	1.32e-3	3.78e-1	4.16e-3
16	3.42e-3	2.64e-4	6.15e-1	3.58e-4
32	2.609e-3	5.41e-4	2.79e-2	5.17e-4
64	4.27e-4	3.92e-5	4.38e-3	3.26e-5
128	8.11e-4	4.59e-6	7.43e-4	2.91e-5

Table *II*

Result of WG finite element method with $\gamma=1/8\pi^2$, $\gamma_\mu=1/4\pi^2$, $a=1$ and $\tau=h^2$.

$1/h$	$\|\nabla(\mathbf{u}-\mathbf{u}_h^n)\|$	$\|\mathbf{u}-\mathbf{u}_h^n\|$	$\|\nabla(\mathbf{B}-\mathbf{B}_h^n)\|$	$\|\mathbf{B}-\mathbf{B}_h^n\|$
8	2.37e-3	1.74e-5	4.05e-2	2.95e-4
16	6.27e-4	3.89e-6	2.71e-2	5.38e-5
32	3.19e-4	7.15e-6	3.83e-3	4.79e-5
64	4.08e-5	2.57e-6	1.82e-4	7.48e-6
128	3.93e-5	5.29e-7	2.36e-5	6.83e-6

From these results, we can know that WG finite element method is advisable and efficient.

Remark 2. For the above exact solutions $\mathbf{u}$ and $\mathbf{B}$, we know that hydrodynamic pressure p and minor control possible electromotive force $\mathbf{J}_c$ are $p = e^{-2t}(\cos^2 2\pi x + \cos^2 2\pi y - \cos 2\pi x \cdot \cos 2\pi y)$, $\mathbf{J}_c = -e^{-2t}(\sin^2 2\pi x \cdot \cos 2\pi y + \cos 2\pi x \cdot \sin^2 2\pi y)$, respectively.

6. Conclusion

In this paper, we have formulated the weak Galerkin finite element scheme for the incompressible viscous magneto-hydrodynamic equations on arbitrary polygons or polyhedra with certain shape regularity.

Also, we have estimated the error of the semi-discrete and full-discrete approximate solutions by the weak Galerkin finite element method for the incompressible viscous Magneto-hydrodynamic equations.

In future work, we will develop for more general problems.

References

[1] M. Gunzburger, A. Meir, J. Peterson, On the existence, uniqueness and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics, Math.Comput. 56(1991) 523–563.

[2] W. Layton, H. Lenferink, J. Peterson, A two-level Newton, finite element algorithm for approximating electrically conducting incompressible fluid flows, Comput. Math. Appl. 28(1994) 21–31.

[3] J. Gerbeau, A stabilized finite element method for the incompressible magnetohydrodynamic equations, Numer. Math.87(2000) 83–111.

[4] N. Salah, A. Soulaimani, W. Habashi, A finite element method for magnetohydrodynamics, Comput. Methods Appl. Mech. Eng.190(2001) 5867–5892.

[5] D. Schötzau, Mixed finite element methods for stationary incompressible magneto-hydrodynamics, Numer. Math. 96(2004) 771–800.

[6] G. Yuksel, R. Ingram, Numerical analysis of a finite element, Crank–Nicolson discretization for MHD flows at small magnetic Reynolds numbers, Int. J. Numer. Anal. Model.10(2013) 74–98.

[7] X. Dong, Y. He, Y. Zhang, Convergence analysis of three finite element iterative methods for the 2D/3D stationary incompressible magnetohydrodynamics, Comput. Methods Appl. Mech. Eng. 276(2014) 287–311.

[8] Y. He, Unconditional convergence of the Euler semi-implicit scheme for the 3D incompressible MHD equations, IMA J. Numer. Anal.35(2) (2015) 767–801.

[9] H. Su, X. Feng, P. Huang, Iterative methods in penalty finite element discretization for the steady MHD equations, Comput. Methods Appl. Mech. Eng. 304(2016) 521–545.

[10] K. Hul, Y. Ma, J. Xu, Stable finite element methods preserving ∇·B=0 exactly for MHD models. Numer. Math. (2017)135: 371–396.

[11] T. Zhu, H. Su, X. Feng, Some Uzawa-type finite element iterative methods for the steady incompressible magneto-hydrodynamic equations. Appl. Math. Comput 302(2017) 34–47.

[12] R. An, C. Zhou, Error analysis of a fractional-step method for magneto-hydrodynamics equations, J. Comput. Appl. Math 313 (2017) 168–184.

[13] J. Wu, D. Liu, X. Feng, P. Huang, An efficient two-step algorithm for the stationary incompressible magneto-hydrodynamic equations. Appl. Math. Comput 302(2017) 21–33.

[14] Z. Sedaghatjoo, M. Dehghan, H. Hosseinzadeh, A stable boundary elements method for magneto-hydrodynamic channel flows at high Hartmann numbers. Numer. Methods Partial Differ. Eq. (2018)34, 575–601.

[15] Y. Rong, Y. Hou, A partitioned finite element scheme based on Gauge-Uzawa method for time-dependent MHD equations. Numer. Algor (2018) 78:277–295.

[16] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comput. Appl. Math. 241 (2013) 103–115.

[17] L. Mu, J. Wang, X. Ye, Weak Galerkin finite element methods on Polytopal Meshes, Int. J. Numer. Anal. Model. 12 (2015) 31–53.

[18] J. Wang, X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp. 83 (2014) 2101–2126.

[19] L. Mu, J. Wang, X. Ye, A weak Galerkin finite element method with polynomial reduction, J. Comput. Appl. Math, 285 (2015) 45–58.

[20] J. Wang, X. Ye, A weak Galerkin finite element method for the Stokes equations, Adv. Comput. Math. 42 (2016) 155–174.

[21] X. Liu, J. Li, Z. Chen, A Weak Galerkin finite element method for the Oseen equations, Adv. Comput. Math. 42 (2016) 1473–1490.

[22] L. Mu, J. Wang, X. Ye, A new weak Galerkin finite element method for the Helmholtz equation, IMA J. Numer. Anal. 35 (2015) 1228–1255.

[23] L. Mu, J. Wang, X. Ye, S. Zhao, A numerical study on the weak Galerkin method for the Helmholtz equation, Commun. Comput. Phys. 15 (2014) 1461–1479.

[24] W. Chen, F. Wang, Y. Wang, Weak Galerkin method for the coupled Darcy-Stokes flow, IMA J. Numer. Anal. 36 (2016) 897–921.

[25] F. Gao, L. Mu, On Error estimate for weak Galerkin finite element methods for parabolic problems, J. Comput. Math. 32 (2014) 195–204.

[26] Q. Li, J. Wang, Weak Galerkin finite element methods for parabolic equations, Numer. Methods Partial Differential Equations 29 (2013) 2004–2024.

[27] F. Gao, X. Wang, A modified weak Galerkin finite element method for a class of parabolic problems, J. Comput. Appl. Math. 271 (2014) 1–19.

[28] S.Y. Yi, A lowest-order weak Galerkin method for linear elasticity, J. Comput. Appl. Math, 350 (2019) 286–298.

[29] R. Li, Y. Gao, J. Li, Z. Chen, A weak Galerkin finite element method for a coupled Stokes–Darcy problem on general meshes, J. Comput. Appl. Math, 334 (2018) 111–127.

[30] Y. Liu, J. Wang, Simplified weak Galerkin and new finite difference schemes for the Stokes equation, J. Comput. Appl. Math, 361 (2019) 176–206.

[31] L. Mu, Weak Galerkin based a posteriori error estimates for second order elliptic interface problems on polygonal meshes, J. Comput. Appl. Math, 361 (2019) 413–425.

[32] G. Chen, M. Feng, A robust WG finite element method for convection–diffusion–reaction equations, J. Comput. Appl. Math, 315 (2017) 107–125.

[33] F. Gao, J. Cui, G. Zhao, Weak Galerkin finite element methods for Sobolev equation, J. Comput. Appl. Math, 317 (2017) 188–202.

[34] R. Temam, Navier-Stokes equations; Theory and Numerical Analysis, North-Holland Publishing Company-Amsterdam, New York, Oxford, 1979.