General Vectorial Lorentz Transformation Formulas for Spacetimes of Arbitrary Dimensions

Tadeusz Pastuszek
Email: tadeuszpastuszek@poczta.fm

Abstract
The article presents generalized vectorial Lorentz transformation formulas applicable to spacetimes of arbitrary dimensions within the framework of special relativity. It introduces a novel notation to differentiate between temporal coordinates and proper time, and assumes the speed of light as dimensionless and set to 1. This approach results in a homogeneous metric space, termed U-space, facilitating the extension of Lorentz transformations beyond the conventional four-dimensional spacetime to spaces with any number of dimensions. The transformations are derived and detailed for velocities, accelerations, and other vectors in U-space, highlighting their universality and ease of application compared to traditional methods.

In the framework of special relativity, we encounter a concept of time that is attributed to two distinct notions:

1. Time as one of the coordinates of spacetime, holding a unique significance. In this context, time as a coordinate possesses spatial properties and is relative, as the Lorentz transformation alters the temporal coordinate values of individual points, just as it does the spatial coordinate values. The time axis in spacetime is typically denoted by the capital letter $\boldsymbol{T}$.

2. Time as the proper time parameter of material points moving through spacetime (denoted by the lowercase letter $\boldsymbol{t}$). The parameter $\boldsymbol{t}$ lacks spatial properties, is a scalar that continuously increases, and remains invariant under Lorentz transformations.

Given these differences between the two approaches, I propose to denote the time axis with the capital letter $\boldsymbol{U}$. Hence, the temporal coordinate of points in spacetime will be indicated by the lowercase letter $\boldsymbol{u}$, while the proper time of material points will continue to be denoted by the letter $\boldsymbol{t}$.

Subsequently, for simplification, we assume that the speed of light is set to 1 and is dimensionless. Therefore, the velocity of massive point objects (vector $\vec{\boldsymbol{v}}$) is also dimensionless, and its absolute value is constrained to the right-open interval $\langle 0\,,1)$. In this scenario, the time axis $\boldsymbol{U}$ must possess a spatial dimension, similar to the other axes. This approach results in a homogeneous metric space, which, to distinguish it from spacetime, we shall refer to as U-space.

Note: In diagrams depicting two-dimensional and three-dimensional spacetimes, it is customary to represent the time axis as vertical. However, in all charts illustrating processes occurring over time, the time axis is always horizontal. Therefore, it appears that representing the time axis as horizontal is more natural and intuitive.

In the context of special relativity, two-dimensional, three-dimensional, and four-dimensional spacetimes (U-spaces) are typically considered, due to the widely accepted view that we live in a four-dimensional spacetime. However, from a formal standpoint, there is nothing that precludes the use of this theory to explore spaces with more dimensions than four. (These dimensions could even number in the millions and beyond). Therefore, let us define an $n+1$-dimensional U-space, where n is any natural number.

The $n+1$-dimensional U-space is a flat, Euclidean, homogeneous, metric space with an imposed Cartesian coordinate system with axes $U, X_1, X_2, ..., , X_n$. The position of points in this space will be denoted by the pair $(u, \vec{r})$, where: $\vec{r} = (x_1, x_2, ..., , x_n)$.

The world line of a point object is defined by the vector function:

$$\vec{r}(u) = [x_1(u), x_2(u), \dots, x_n(u)] \tag{1}$$

The velocity $\vec{v}$ and acceleration $\vec{a}$ of this object are also vector functions:

$$\vec{v}(u) = \frac{d\vec{r}(u)}{du} = \left[\frac{dx_1(u)}{du}, \frac{dx_2(u)}{du}, \dots, \frac{dx_n(u)}{du}\right] \tag{2}$$

$$\vec{a}(u) = \frac{d\vec{v}(u)}{du} = \frac{d^2\vec{r}(u)}{du^2} = \left[\frac{d^2x_1(u)}{du^2}, \frac{d^2x_2(u)}{du^2}, \dots, \frac{d^2x_n(u)}{du^2}\right] \tag{3}$$

The velocity vector $\vec{v} = (v_1, v_2, ..., , v_n)$ is subject to a constraint on its magnitude $|\vec{v}| < 1$, meaning:

$$\sqrt{\sum_{i=1}^{n} v_i^2} < 1 \tag{4}$$

Therefore, the velocity vector space is nonlinear, confined to the interior of an $\boldsymbol{n}$-dimensional sphere with radius 1, where the composition of two velocities is governed by a nonlinear formula:

$$\vec{v} = \frac{\sqrt{1 - \left|\vec{V}\right|^2}}{1 + \vec{v_0} \cdot \vec{V}} \vec{v_0} + \frac{\vec{v_0} \cdot \vec{V} + 1 + \sqrt{1 - \left|\vec{V}\right|^2}}{\left(1 + \vec{v_0} \cdot \vec{V}\right)\left(1 + \sqrt{1 - \left|\vec{V}\right|^2}\right)} \vec{V} \tag{5}$$

where $\vec{v}_0$ is the velocity of an object in the reference frame of another object, whose velocity is $\vec{V}$. The dot denotes the scalar product of two vectors. The above formula results from the Lorentz transformation and applies to U-space of any number of dimensions.

Note! Formula (5) was derived in the appendix at the end of the article as formula (A.40). The above formula is more commonly known in the version pertaining to two-dimensional U-space:

$$v = \frac{V + v_0}{1 + V v_0} \tag{6}$$

In turn, the proper time of a massive point object is determined by the formula:

$$t(u) = \int_{u_0}^{u} \sqrt{1 - |\vec{v}(u)|^2}\, du \tag{7}$$

u_0 is the location where the world line of a point object begins. We observe that the proper time of a point object is a positive function that continuously increases, indicating that the object's time is its scalar parameter greater than or equal to zero. This formula is a result of the Special Theory of Relativity (STR). The proper time of point objects in U-space still measures distance. The object's time literally defines the length of its world line measured in its reference frame. Physically, time always has a non-negative value and continually increases for individual massive physical points moving in U-space. A second is simply a measure of distance with the conversion factor $c = 299\,792\,458\ [m/s]$. It is with this speed that we move along the U-axis in U-space.

In the concept of U-space, the time axis $\boldsymbol{T}$ in spacetime, which is scaled in seconds with the scale factor $\boldsymbol{c}$, has been replaced by the U-axis, which has the same measure as the other axes. We have obtained a homogeneous, flat, metric space with an imposed Cartesian coordinate system. Therefore, we must adapt the Lorentz transformation to these new notations.

To begin, let us define the Lorentz transformation vector. The non-zero Lorentz transformation vector will be denoted by the symbol $\vec{\boldsymbol{w}}$. This vector specifies the velocity of an object for which we wish to rotate the axis $\boldsymbol{U}$ such that, after the transformation, the object moving at such velocity in U-space would have its world line (or tangent to the world line in cases where the line is not straight) parallel to the axis $\boldsymbol{U}$. Simply put, by using the Lorentz transformation, we aim to transition into the rest frame of reference of the object moving at the velocity $\vec{\boldsymbol{w}}$.

However, we cannot accomplish this through a mere isometric rotation of U-space, because, after the transformation, light must still travel at speed 1 (the light ray must still form a 45° angle with respect to the axis $\boldsymbol{U}$). After the transformation, within the "field of view" (in the light cone directed towards the past) of each observer, all objects that were visible before the transformation must remain visible. However, the coordinates $\boldsymbol{u}$ and $\vec{\boldsymbol{r}}$ of points' positions in U-space, the velocity vectors $\vec{\boldsymbol{v}}$, and the acceleration vectors $\vec{\boldsymbol{a}}$ of all massive point objects on their world lines will undergo changes. The total energies $\boldsymbol{E_c}$ of massive objects as well as the kinetic energies of light rays will also change. The wavelength of light rays (photon) will be denoted by $\boldsymbol{\lambda}$. Meanwhile, the vector connecting the observed object in the light cone to the observer will be denoted as $\vec{\boldsymbol{s}}$. The unit vector $\hat{\boldsymbol{s}}$ will indicate the direction of the light ray sent by the observed object.

The aforementioned elements, after transformation, will be denoted with a prime symbol ($'$). To begin, we will present the Lorentz transformation formulas for two-dimensional U-space, which are significantly simpler than those for spaces of higher dimensions. In two-dimensional U-space, all mentioned vectors are one-dimensional, hence the vector notation is omitted. For two-dimensional U-space, the Lorentz transformation is described by the formulas:

$$\boldsymbol{u}' = \frac{\boldsymbol{u} - \boldsymbol{x}\boldsymbol{w}}{\sqrt{1 - \boldsymbol{w}^2}} \tag{8}$$

$$\boldsymbol{x}' = \frac{\boldsymbol{x} - \boldsymbol{u}\boldsymbol{w}}{\sqrt{1 - \boldsymbol{w}^2}} \tag{9}$$

$$\boldsymbol{v}' = \frac{\boldsymbol{v} - \boldsymbol{w}}{1 - \boldsymbol{v}\boldsymbol{w}} \tag{10}$$

$$\boldsymbol{a}' = \boldsymbol{a}\left(\frac{\sqrt{1 - \boldsymbol{w}^2}}{1 - \boldsymbol{v}\boldsymbol{w}}\right)^3 \tag{11}$$

$$\boldsymbol{s}' = \boldsymbol{s}\frac{1 - \hat{\boldsymbol{s}}\boldsymbol{w}}{\sqrt{1 - \boldsymbol{w}^2}} \tag{12}$$

$$\hat{\boldsymbol{s}}' = \hat{\boldsymbol{s}} \tag{13}$$

$$\boldsymbol{\lambda}' = \boldsymbol{\lambda}\frac{\sqrt{1 - \boldsymbol{w}^2}}{1 - \hat{\boldsymbol{s}}\boldsymbol{w}} \tag{14}$$

$$\boldsymbol{E}_c' = \boldsymbol{E}_c\frac{1 - \boldsymbol{v}\boldsymbol{w}}{\sqrt{1 - \boldsymbol{w}^2}} \tag{15}$$

For U-space with more dimensions than two, the Lorentz transformation formulas are somewhat more complex:

$$\boldsymbol{u}' = \frac{\boldsymbol{u} - \vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{w}}}{\sqrt{1 - |\vec{\boldsymbol{w}}|^2}} \tag{16}$$

$$\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{r}} + \left[\frac{\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{w}}}{\sqrt{1-|\vec{\boldsymbol{w}}|^2}\left(1+\sqrt{1-|\vec{\boldsymbol{w}}|^2}\right)} - \frac{\boldsymbol{u}}{\sqrt{1-|\vec{\boldsymbol{w}}|^2}}\right]\vec{\boldsymbol{w}} \tag{17}$$

$$\vec{\boldsymbol{v}}\,' = \frac{\sqrt{1 - |\vec{\boldsymbol{w}}|^2}}{1 - \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}\vec{\boldsymbol{v}} + \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}} - 1 - \sqrt{1 - |\vec{\boldsymbol{w}}|^2}}{(1 - \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}})\left(1 + \sqrt{1 - |\vec{\boldsymbol{w}}|^2}\right)}\vec{\boldsymbol{w}} \tag{18}$$

$$\vec{a}\,' = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\vec{a}+\frac{\vec{a}\cdot\vec{w}}{1-\vec{v}\cdot\vec{w}}\vec{v}-\frac{\vec{a}\cdot\vec{w}}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w}\right] \tag{19}$$

$$\vec{s}\,' = \vec{s}-\frac{|\vec{s}|\left(1-\hat{s}\cdot\vec{w}+\sqrt{1-|\vec{w}|^2}\right)}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w} \tag{20}$$

$$\hat{s}' = \frac{\sqrt{1-|\vec{w}|^2}}{1-\hat{s}\cdot\vec{w}}\hat{s}-\frac{1-\hat{s}\cdot\vec{w}+\sqrt{1-|\vec{w}|^2}}{(1-\hat{s}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w} \tag{21}$$

$$\lambda' = \lambda\frac{\sqrt{1-|\vec{w}|^2}}{1-\hat{s}\cdot\vec{w}} \tag{22}$$

$$E_c' = E_c\frac{1-\vec{v}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}} \tag{23}$$

Based on the Lorentz transformation formulas, it can be readily verified that the proper time of an object is an invariant of the Lorentz transformation. If we calculate the integral (7) for any two points located on the same world line, the value of this integral in all reference frames remains the same. Therefore, the time of massive physical objects is not relative; it is an objective category, though it is distinct for each physical point. The time of a point object precisely determines its position on the world line.

Appendix

The general Lorentz transformation formulas from (16) to (21), as presented above, are not found anywhere in the scientific literature, hence the necessity to provide derivations. These formulas will first be derived for three-dimensional U-space. Subsequently, they will be generalized to pertain to U-space of any number of dimensions.

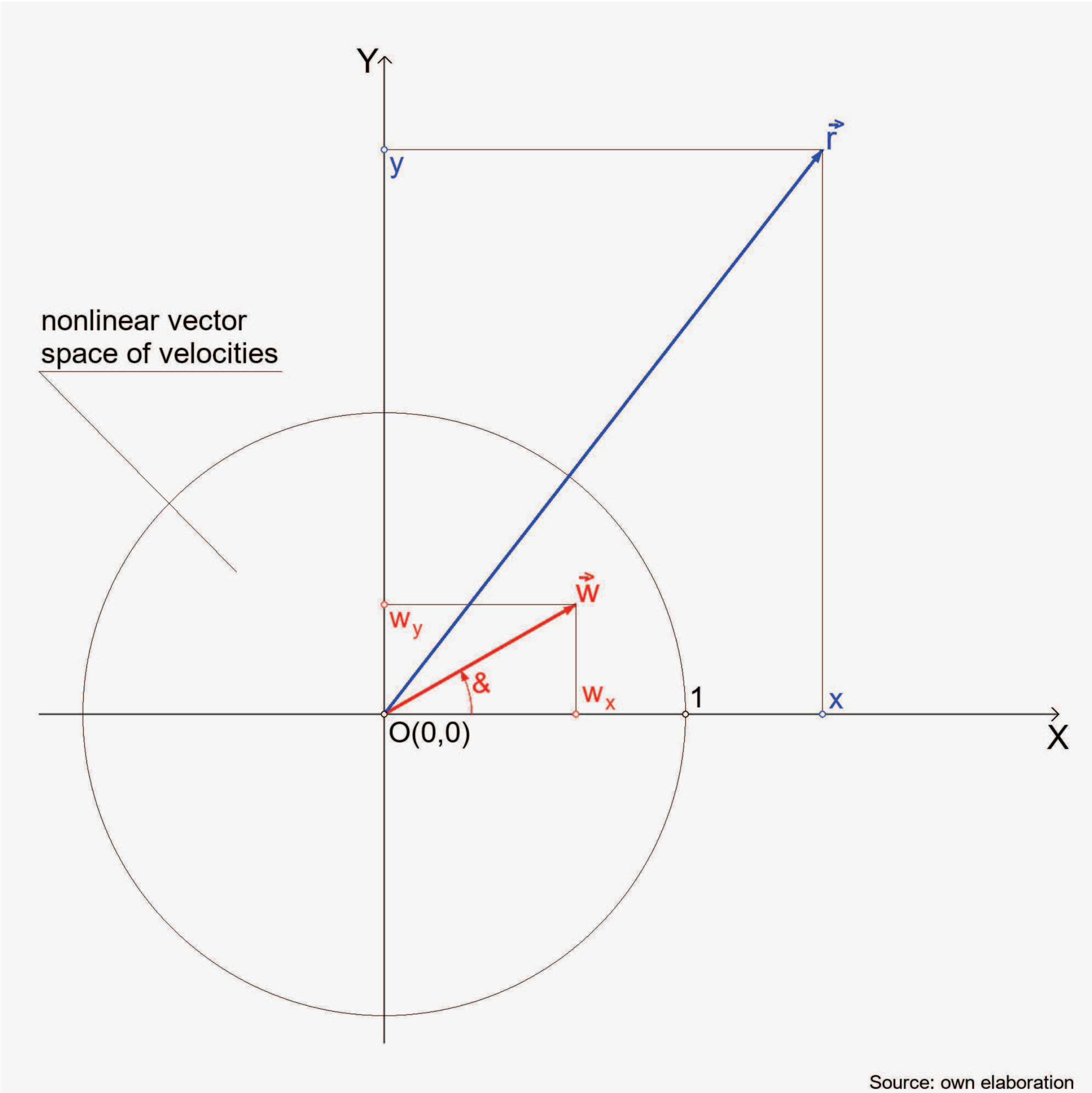

Figure 1

Above in Figure 1, a plane with a Cartesian coordinate system with axes $\boldsymbol{X}$, $\boldsymbol{Y}$ imposed on it is depicted. This plane is a section of the U-space at the point $\boldsymbol{u}$ and is perpendicular to the U-axis. On this plane lie vectors $\vec{w}$ and $\vec{r}$, where vector $\vec{w}$ is a Lorentz transformation vector representing the velocity of the object for which we want to determine the "rest" frame of reference. Any point on this plane is marked as vector $\vec{r}$. Figure 1 also shows a nonlinear velocity vector space, which is bounded within a circle of radius 1.

We will begin by deriving the formula for the Lorentz transformation using the vector $\vec{w}$ for the coordinate u and the vector $\vec{r}$. To this end, we will rotate the axes X, Y around the axis U by an angle &. The axes in this new position will be denoted X_1, Y_1. The relationships between the coordinates (x, y) of vectors in the original coordinate system and the coordinates (x_1, y_1) of vectors in the new axis system are as follows:

$$x_1 = x\cos\& + y\sin\& \tag{A.1}$$

$$y_1 = y\cos\& - x\sin\& \tag{A.2}$$

where:

$$\sin\& = \frac{w_y}{|\vec{w}|}, \quad \cos\& = \frac{w_x}{|\vec{w}|} \tag{A.3}$$

Based on the above formulas, we obtain:

$$w_{x_1} = |\vec{w}|, \quad w_{y_1} = 0 \tag{A.4}$$

$$x_1 = \frac{\vec{r}\cdot\vec{w}}{|\vec{w}|}, \quad y_1 = \frac{yw_x - xw_y}{|\vec{w}|} \tag{A.5}$$

where the expression with a dot denotes the scalar product of vectors.

Now, based on formula (8), we will calculate u':

$$u' = \frac{u - x_1 w_{x_1}}{\sqrt{1 - |\vec{w}|^2}}$$

By substituting the expressions from (A.4) and (A.5), we obtain:

$$u' = \frac{u - \vec{r}\cdot\vec{w}}{\sqrt{1 - |\vec{w}|^2}} \tag{A.6}$$

Then, based on formula (9), we will calculate x_1':

$$x_1' = \frac{x_1 - u w_{x_1}}{\sqrt{1 - |\vec{w}|^2}}$$

Similarly to before, we substitute the expressions from (A.4) and (A.5):

$$\boldsymbol{x}_1' = \frac{\frac{\vec{r}\cdot\vec{w}}{|\vec{w}|} - u|\vec{w}|}{\sqrt{1-|\vec{w}|^2}} \tag{A.7}$$

The value of the coordinate $\boldsymbol{y_1}$ after transformation will not change, because the axis $\boldsymbol{Y_1}$ is perpendicular to the transformation vector $\vec{w}$.

$$y_1' = y_1 = \frac{yw_x - xw_y}{|\vec{w}|} \tag{A.8}$$

To return to the coordinate system with axes $\boldsymbol{X}$, $\boldsymbol{Y}$, we must rotate the axes $\boldsymbol{X_1}$, $\boldsymbol{Y_1}$ around $\boldsymbol{O}$ by an angle $-\&$.

$$x' = x_1' \cos\& - y_1' \sin\& \tag{A.9}$$

$$y' = x_1' \sin\& + y_1' \cos\& \tag{A.10}$$

When we substitute the expressions from (A.7) and (A.8) as well as from (A.3) into (A.9) and (A.10), we obtain:

$$x' = \frac{\vec{r}\cdot\vec{w} - u|\vec{w}|^2}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} w_x - \frac{yw_x - xw_y}{|\vec{w}|^2} w_y \tag{A.11}$$

$$y' = \frac{\vec{r}\cdot\vec{w} - u|\vec{w}|^2}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} w_y - \frac{xw_y - yw_x}{|\vec{w}|^2} w_x \tag{A.12}$$

Formulas (A.11) and (A.12), which are symmetric with respect to the coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$, need to be brought into a form that allows them to be expressed by a single vector equation (17).

$$x' = \left(\frac{xw_x + yw_y}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right) w_x - \frac{yw_y w_x}{|\vec{w}|^2} + \frac{xw_y^2}{|\vec{w}|^2}$$

$$x' = \frac{xw_y^2}{|\vec{w}|^2} + \left(\frac{xw_x + yw_y}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} - \frac{u}{\sqrt{1-|\vec{w}|^2}} - \frac{yw_y}{|\vec{w}|^2}\right) w_x$$

$$x' = \frac{xw_y^2}{|\vec{w}|^2} + \left[\frac{xw_x}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} + \frac{yw_y\left(1 - \sqrt{1-|\vec{w}|^2}\right)}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right] w_x$$

$$Note:\ \frac{1-\sqrt{1-|\vec{w}|^2}}{|\vec{w}|^2} = \frac{1}{1+\sqrt{1-|\vec{w}|^2}}$$

$$x' = \frac{xw_y^2}{|\vec{w}|^2} + \left[\frac{xw_x}{|\vec{w}|^2\sqrt{1-|\vec{w}|^2}} + \frac{yw_y}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]w_x$$

$$Note:\ \sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right) = 1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2$$

$$x' = \frac{xw_y^2}{|\vec{w}|^2} + \left[\frac{xw_x\left(1+\sqrt{1-|\vec{w}|^2}\right)}{|\vec{w}|^2\left(1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2\right)} + \frac{yw_y + (xw_x - xw_x)}{1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]w_x$$

$$x' = \frac{xw_y^2}{|\vec{w}|^2} + \left[\frac{xw_x\left(1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2\right)}{|\vec{w}|^2\left(1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2\right)} + \frac{yw_y + xw_x}{1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]w_x$$

$$x' = \frac{xw_y^2 + xw_x^2}{|\vec{w}|^2} + \left(\frac{\vec{r}\cdot\vec{w}}{1+\sqrt{1-|\vec{w}|^2} - |\vec{w}|^2} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right)w_x$$

$$x' = x + \left[\frac{\vec{r}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]w_x \qquad \text{(A.13)}$$

Due to the symmetry between the coordinates x, y, the transformation of formula (A.12) will look analogous. Therefore, we can state without performing the calculations:

$$y' = y + \left[\frac{\vec{r}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]w_y \qquad \text{(A.14)}$$

Formulas (A.13) and (A.14) for three-dimensional U-space are the decomposition into coordinates of the vector $\vec{r}\,'$ of the general vector formula (17):

$$\vec{r}\,' = \vec{r} + \left[\frac{\vec{r}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]\vec{w} \qquad \text{(A.15)}$$

It should be noted that if we apply the reverse Lorentz transformation vector $-\vec{w}$ to the coordinates (u', x', y'), after the transformation of some point, by using formulas (A.6) and (A.15), we will obtain the original coordinates $(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y})$ of that point. It is easy to verify that the general formulas (A.6) and (A.15) apply to U-space of any number of dimensions.

To derive formula (18) for the transformation of the velocity vector, we will use formulas (A.6) and (A.15), which were derived above. We assume that an inertial point with velocity $\vec{\boldsymbol{v}}$ starts from the origin of the coordinate system of U-space, of any number of dimensions. Adopting this assumption will significantly simplify the calculations, as the origin of the U-space coordinate system does not change its position when applying the Lorentz transformation. The position $\vec{\boldsymbol{r}}$ of this point as a function of $\boldsymbol{u}$ is defined by the formula: $\vec{\boldsymbol{r}} = \boldsymbol{u}\vec{\boldsymbol{v}}$. Meanwhile, after transformation with the vector $\vec{\boldsymbol{w}}$, we have the formula: $\vec{\boldsymbol{r}}\,' = \boldsymbol{u}'\vec{\boldsymbol{v}},'$, which means:

$$\vec{v}\,' = \frac{1}{u'}\,\vec{r}\,' \tag{A.16}$$

Let us substitute into the above formula the expressions from (A.6) and (A.15):

$$\vec{v}\,' = \frac{\sqrt{1-|\vec{w}|^2}}{u - \vec{r}\cdot\vec{w}}\vec{r} + \frac{\sqrt{1-|\vec{w}|^2}}{u - \vec{r}\cdot\vec{w}}\left[\frac{\vec{r}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{u}{\sqrt{1-|\vec{w}|^2}}\right]\vec{w}$$

For the sake of simplifying the calculations, we assume the value $\boldsymbol{u} = \mathbf{1}$, then $\vec{\boldsymbol{r}} = \vec{\boldsymbol{v}}$ and we can write:

$$\vec{v}\,' = \frac{\sqrt{1-|\vec{w}|^2}}{1 - \vec{v}\cdot\vec{w}}\vec{v} + \frac{\sqrt{1-|\vec{w}|^2}}{1 - \vec{v}\cdot\vec{w}}\left[\frac{\vec{v}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{1}{\sqrt{1-|\vec{w}|^2}}\right]\vec{w}$$

$$\vec{v}\,' = \frac{\sqrt{1-|\vec{w}|^2}}{1 - \vec{v}\cdot\vec{w}}\vec{v} + \left[\frac{\vec{v}\cdot\vec{w}}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{1}{1-\vec{v}\cdot\vec{w}}\right]\vec{w}$$

$$\vec{v}\,' = \frac{\sqrt{1-|\vec{w}|^2}}{1 - \vec{v}\cdot\vec{w}}\vec{v} + \frac{\vec{v}\cdot\vec{w} - 1 - \sqrt{1-|\vec{w}|^2}}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w} \tag{A.17}$$

Thus, we have obtained the general formula (18). This formula is applicable to U-space of any number of dimensions. Also, by transforming

$\vec{v}\,'$ with the reverse vector $-\vec{w}$, we will retrieve the original velocity vector $\vec{v}$.

To derive formula (19) for the transformation of the acceleration vector $\vec{a}$, we will use formulas (A.1) and (A.2) for the rotation of axes X, Y in order to align the Lorentz transformation vector $\vec{w}$ with axis X_1. After the rotation, we obtain the following vectors:

$$w_{x_1} = |\vec{w}|\,, \qquad w_{y_1} = 0 \tag{A.18}$$

$$v_{x_1} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}, \qquad v_{y_1} = \frac{v_y w_x - v_x w_y}{|\vec{w}|} \tag{A.19}$$

$$a_{x_1} = \frac{\vec{a}\cdot\vec{w}}{|\vec{w}|}, \qquad a_{y_1} = \frac{a_y w_x - a_x w_y}{|\vec{w}|} \tag{A.20}$$

From the generally known Lorentz transformation formulas for a two-dimensional acceleration vector, we have:

$$a'_{x_1} = \left(\frac{\sqrt{1-|\vec{w}|^2}}{1-|\vec{w}|v_{x_1}}\right)^3 a_{x_1} \tag{A.21}$$

$$a'_{y_1} = \frac{1-|\vec{w}|^2}{(1-|\vec{w}|v_{x_1})^2}\left(a_{y_1} + \frac{|\vec{w}|v_{y_1}}{1-|\vec{w}|v_{x_1}}a_{x_1}\right) \tag{A.22}$$

Upon substituting the expressions from formulas (A.19) and (A.20):

$$a'_{x_1} = \frac{\vec{a}\cdot\vec{w}\left(\sqrt{1-|\vec{w}|^2}\right)^3}{|\vec{w}|\,(1-\vec{v}\cdot\vec{w})^3} \tag{A.23}$$

$$a'_{y_1} = \frac{1-|\vec{w}|^2}{|\vec{w}|\,(1-\vec{v}\cdot\vec{w})^2}\left[a_y w_x - a_x w_y + \frac{(v_y w_x - v_x w_y)\,\vec{a}\cdot\vec{w}}{1-\vec{v}\cdot\vec{w}}\right]$$

$$a'_{y_1} = \frac{1-|\vec{w}|^2}{|\vec{w}|\,(1-\vec{v}\cdot\vec{w})^2}\left[\frac{(a_y w_x - a_x w_y)(1-v_x w_x - v_y w_y)+(v_y w_x - v_x w_y)(a_x w_x + a_y w_y)}{1-\vec{v}\cdot\vec{w}}\right]$$

$$a'_{y_1} = \frac{1-|\vec{w}|^2}{|\vec{w}|\,(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_y w_x - a_x w_y + (a_x v_y - a_y v_x)|\vec{w}|^2}{1-\vec{v}\cdot\vec{w}}\right]$$

$$a'_{y_1} = \frac{1-|\vec{w}|^2}{|\vec{w}|\,(1-\vec{v}\cdot\vec{w})^3}\left[a_y w_x - a_x w_y + (a_x v_y - a_y v_x)|\vec{w}|^2\right] \tag{A.24}$$

To return to the previous coordinate system $\boldsymbol{X}$, $\boldsymbol{Y}$, we will apply formulas (A.9) and (A.10) along with (A.3):

$$\boldsymbol{a'_x = a'_{x_1}\frac{w_x}{|\vec{w}|} - a'_{y_1}\frac{w_y}{|\vec{w}|}} \tag{A.25}$$

$$\boldsymbol{a'_y = a'_{x_1}\frac{w_y}{|\vec{w}|} + a'_{y_1}\frac{w_x}{|\vec{w}|}} \tag{A.26}$$

After substituting expressions from (A.23) and (A.24) into (A.25) and (A.26), we obtain:

$$\boldsymbol{a'_x = \frac{w_x\,(\vec{a}\cdot\vec{w})\left(\sqrt{1-|\vec{w}|^2}\right)^3 + w_y(1-|\vec{w}|^2)[a_x w_y - a_y w_x + (a_y v_x - a_x v_y)|\vec{w}|^2]}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})^3}} \tag{A.27}$$

$$\boldsymbol{a'_y = \frac{w_y\,(\vec{a}\cdot\vec{w})\left(\sqrt{1-|\vec{w}|^2}\right)^3 + w_x(1-|\vec{w}|^2)[a_y w_x - a_x w_y + (a_x v_y - a_y v_x)|\vec{w}|^2]}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})^3}} \tag{A.28}$$

Now, formulas (A.27) and (A.28) need to be transformed in such a way that they can be written in the form of a single, general formula (19).

$$\boldsymbol{a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{w_x\,(\vec{a}\cdot\vec{w})\sqrt{1-|\vec{w}|^2} + w_y(a_x w_y - a_y w_x)}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{w_y(a_y v_x - a_x v_y)}{1-\vec{v}\cdot\vec{w}}\right]}$$

$$\boldsymbol{a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{(a_x w_x^2 + a_y w_x w_y)\sqrt{1-|\vec{w}|^2} + a_x w_y^2 - a_y w_x w_y}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{w_y(a_y v_x - a_x v_y)}{1-\vec{v}\cdot\vec{w}}\right]}$$

$$\boldsymbol{a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_x w_x^2\sqrt{1-|\vec{w}|^2} + a_y w_x w_y\left(\sqrt{1-|\vec{w}|^2}-1\right) + a_x w_y^2}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{w_y(a_y v_x - a_x v_y)}{1-\vec{v}\cdot\vec{w}}\right]}$$

$$\boldsymbol{a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_x w_x^2\sqrt{1-|\vec{w}|^2} + a_y w_x w_y\left(\sqrt{1-|\vec{w}|^2}-1\right) + a_x w_y^2 + (a_x w_x^2 - a_x w_x^2)}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{w_y(a_y v_x - a_x v_y)}{1-\vec{v}\cdot\vec{w}}\right]}$$

$$\boldsymbol{a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_x w_x^2\left(\sqrt{1-|\vec{w}|^2}-1\right) + a_y w_x w_y\left(\sqrt{1-|\vec{w}|^2}-1\right) + a_x(w_y^2 + w_x^2)}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{w_y(a_y v_x - a_x v_y)}{1-\vec{v}\cdot\vec{w}}\right]}$$

$$\boldsymbol{a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{(a_x w_x^2 + a_y w_x w_y)\left(\sqrt{1-|\vec{w}|^2}-1\right) + a_x|\vec{w}|^2}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{w_y(a_y v_x - a_x v_y)}{1-\vec{v}\cdot\vec{w}}\right]}$$

$$a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_x}{1-\vec{v}\cdot\vec{w}} - \frac{w_x(a_x w_x + a_y w_y)\left(1-\sqrt{1-|\vec{w}|^2}\right)}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{a_y w_y v_x - a_x w_y v_y}{1-\vec{v}\cdot\vec{w}}\right]$$

$$a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_x}{1-\vec{v}\cdot\vec{w}} - \frac{w_x(\vec{a}\cdot\vec{w})\left(1-\sqrt{1-|\vec{w}|^2}\right)}{|\vec{w}|^2(1-\vec{v}\cdot\vec{w})} + \frac{a_y w_y v_x - a_x w_y v_y + (a_x w_x v_x - a_x w_x v_x)}{1-\vec{v}\cdot\vec{w}}\right]$$

$$Note:\ \frac{1-\sqrt{1-|\vec{w}|^2}}{|\vec{w}|^2} = \frac{1}{1+\sqrt{1-|\vec{w}|^2}}$$

$$a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\frac{a_x}{1-\vec{v}\cdot\vec{w}} - \frac{w_x(\vec{a}\cdot\vec{w})}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)} + \frac{v_x(\vec{a}\cdot\vec{w}) - a_x(\vec{v}\cdot\vec{w})}{1-\vec{v}\cdot\vec{w}}\right]$$

$$a'_x = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[a_x + \frac{\vec{a}\cdot\vec{w}}{1-\vec{v}\cdot\vec{w}}v_x - \frac{\vec{a}\cdot\vec{w}}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)}w_x\right] \tag{A.29}$$

Analogously to (A.27), we transform formula (A.28):

$$a'_y = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[a_y + \frac{\vec{a}\cdot\vec{w}}{1-\vec{v}\cdot\vec{w}}v_y - \frac{\vec{a}\cdot\vec{w}}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)}w_y\right] \tag{A.30}$$

Formulas (A.29) and (A.30) represent the decomposition into coordinates a'_x and a'_y of the general formula (19) for three-dimensional U-space:

$$\vec{a}\,' = \frac{1-|\vec{w}|^2}{(1-\vec{v}\cdot\vec{w})^2}\left[\vec{a} + \frac{\vec{a}\cdot\vec{w}}{1-\vec{v}\cdot\vec{w}}\vec{v} - \frac{\vec{a}\cdot\vec{w}}{(1-\vec{v}\cdot\vec{w})\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w}\right] \tag{A.31}$$

Formula (A.31) applies to U-space of any dimension. When we apply formula (A.31) to the vector $\vec{a}\,'$ with the inverse Lorentz transformation vector $-\vec{w}$, we obtain the original acceleration vector $\vec{a}$.

We will derive formula (20) for the transformation of vector $\vec{s}$ immediately for U-space of any dimension (similarly to what we did for the velocity vector). We assume that the vector $\vec{s} = \vec{r}_B - \vec{r}_A$, where point

$\boldsymbol{A}$ is located in the light cone of observer $\boldsymbol{B}$, so $|\vec{s}| = u_B - u_A$. For simplicity of calculations, we assume that point $\boldsymbol{A}$ is located at the origin of the coordinate system $\boldsymbol{O}$, then:

$$\vec{r}_B = \vec{s}, \qquad u_B = |\vec{s}| \tag{A.32}$$

With this assumption, we can employ formula (A.15) for the transformation of vector $\vec{s}$, substituting the values from (A.32) into it:

$$\vec{s}\,' = \vec{s} + \left[\frac{\vec{s}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)} - \frac{|\vec{s}|}{\sqrt{1-|\vec{w}|^2}}\right]\vec{w} \tag{A.33}$$

$$\vec{s}\,' = \vec{s} + \frac{|\vec{s}|(\hat{s}\cdot\vec{w}) - |\vec{s}|\left(1+\sqrt{1-|\vec{w}|^2}\right)}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w} \tag{A.34}$$

$$\vec{s}\,' = \vec{s} - \frac{|\vec{s}|\left(1-\hat{s}\cdot\vec{w}+\sqrt{1-|\vec{w}|^2}\right)}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w} \tag{A.35}$$

Thus, we obtained formula (20). Formula (21) for the unit vector $\hat{s}'$ is derived by dividing the right side of equation (A.35) by the length of vector $|\vec{s}\,'|$. From (A.32) and the assumptions made during the derivation of (A.35), it follows that $u_B = |\vec{s}|$ and consequently $u'_B = |\vec{s}\,'|$. Therefore, from (A.6) we have:

$$|\vec{s}\,'| = u'_B = \frac{u_B - \vec{r}_B\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}} \tag{A.36}$$

Upon substituting the expressions from (A.32), we obtain:

$$|\vec{s}\,'| = \frac{|\vec{s}| - \vec{s}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}} \tag{A.37}$$

$$|\vec{s}\,'| = |\vec{s}|\frac{1-\hat{s}\cdot\vec{w}}{\sqrt{1-|\vec{w}|^2}} \tag{A.38}$$

To obtain equation (21) for $\hat{s}'$, we divide (A.35) by (A.38):

$$\hat{s}' = \frac{\sqrt{1-|\vec{w}|^2}}{(1-\hat{s}\cdot\vec{w})|\vec{s}|}\vec{s} - \frac{\sqrt{1-|\vec{w}|^2}}{(1-\hat{s}\cdot\vec{w})|\vec{s}|}\,\frac{|\vec{s}|\left(1-\hat{s}\cdot\vec{w}+\sqrt{1-|\vec{w}|^2}\right)}{\sqrt{1-|\vec{w}|^2}\left(1+\sqrt{1-|\vec{w}|^2}\right)}\vec{w}$$

$$\hat{\boldsymbol{s}}' = \frac{\sqrt{\mathbf{1} - |\vec{\boldsymbol{w}}|^2}}{\mathbf{1} - \hat{\boldsymbol{s}} \cdot \vec{\boldsymbol{w}}} \hat{\boldsymbol{s}} - \frac{\mathbf{1} - \hat{\boldsymbol{s}} \cdot \vec{\boldsymbol{w}} + \sqrt{\mathbf{1} - |\vec{\boldsymbol{w}}|^2}}{(\mathbf{1} - \hat{\boldsymbol{s}} \cdot \vec{\boldsymbol{w}})\left(\mathbf{1} + \sqrt{\mathbf{1} - |\vec{\boldsymbol{w}}|^2}\right)} \vec{\boldsymbol{w}} \tag{A.39}$$

The derived formulas (16) - (21) of Lorentz transformation appear to be more universal and easier to apply than the methods previously used. So far, to perform the Lorentz transformation, the coordinate system was rotated to align the transformation vector, denoted here as $\vec{\boldsymbol{w}}$, with one of the axes of the rotated coordinate system. (Typically, this was the $\boldsymbol{X}$ axis). Therefore, each coordinate had to be treated separately, and consequently, general transformations for spacetime of any dimensionality could not be conducted. (For example, four-vectors only applied to four-dimensional spacetime). It is essential to emphasize this: the Special Theory of Relativity imposes no restrictions on the number of dimensions of spacetime (U-space).

It remains to show where the nonlinear formula (5) for adding two velocities came from. It provides, in a given reference frame, the velocity $\vec{\boldsymbol{v}}$ of an object that, in the reference frame of another object moving with velocity $\vec{\boldsymbol{V}}$, has velocity $\vec{\boldsymbol{v}}_{\mathbf{0}}$. Formula (5) directly arises from formula (A.17), where we must subject the vector $\vec{\boldsymbol{v}}_{\mathbf{0}}$ to Lorentz transformation by the vector $\vec{\boldsymbol{w}} = -\vec{\boldsymbol{V}}$:

$$\vec{\boldsymbol{v}} = \frac{\sqrt{\mathbf{1} + |\vec{\boldsymbol{V}}|^2}}{\mathbf{1} - \vec{\boldsymbol{v}}_0 \cdot (-\vec{\boldsymbol{V}})} \vec{\boldsymbol{v}}_0 + \frac{\vec{\boldsymbol{v}}_0 \cdot (-\vec{\boldsymbol{V}}) - \mathbf{1} - \sqrt{\mathbf{1} - |\vec{\boldsymbol{V}}|^2}}{\left[\mathbf{1} - \vec{\boldsymbol{v}}_0 \cdot (-\vec{\boldsymbol{V}})\right]\left(\mathbf{1} + \sqrt{\mathbf{1} - |\vec{\boldsymbol{V}}|^2}\right)} (-\vec{\boldsymbol{V}})$$

$$\vec{\boldsymbol{v}} = \frac{\sqrt{\mathbf{1} - |\vec{\boldsymbol{V}}|^2}}{\mathbf{1} + \vec{\boldsymbol{v}}_0 \cdot \vec{\boldsymbol{V}}} \vec{\boldsymbol{v}}_0 + \frac{\vec{\boldsymbol{v}}_0 \cdot \vec{\boldsymbol{V}} + \mathbf{1} + \sqrt{\mathbf{1} - |\vec{\boldsymbol{V}}|^2}}{\left(\mathbf{1} + \vec{\boldsymbol{v}}_0 \cdot \vec{\boldsymbol{V}}\right)\left(\mathbf{1} + \sqrt{\mathbf{1} - |\vec{\boldsymbol{V}}|^2}\right)} \vec{\boldsymbol{V}} \tag{A.40}$$