The Time-Neutral Exterior Schwarzschild Metric in Polynomial Form

Kevin Loch*

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The time-neutral metric is introduced and the time-neutral exterior Schwarzschild metric is converted to polynomial form in r and total mass M. The polynomials are found to be cubic with no constant term, which allows the two non-zero roots of each to be extracted from the reduced quadratic form.

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THE TIME-NEUTRAL METRIC

A time-neutral metric is an exact solution to the Einstein field equations[1] multiplied by the inverse length scalar $1/c^2dt^2$, and set equal to one. The time-neutral metric with signature $(+,-,-,-)$ is given by:

$$\frac{ds^2}{c^2dt^2} = \frac{d\tau^2}{dt^2} = 1. \tag{1}$$

Multiplying the line element by this scalar is a special coordinate transformation that also converts coordinate infinitessimals to dimensionless ratios[1]

Setting the transformed metric to one represents neutral time dilation ($\Delta t' = \Delta t$), as opposed to maximal time dilation of the zero-time metric $d\tau^2/dt^2 = 0$. That finite, real, solutions to time-neutral metrics exist with $M \neq 0$ may be surprising. It is especially counter-intuitive for the Schwarzschild metric[2], which does not have the opposite sign r_Q^2 or a^2 factors that give additional degrees of freedom[3, 4] to higher order metrics. However, these solutions do exist mathematically.

Further transforming the time-neutral metric into polynomial form has the effect of replacing the divergence at $r = 0$ in the standard form of the metric, with the constraint $v_r^2/c^2 \neq 0$ in the polynomial form. Even if this is not physical it might be a useful tool for quantum gravity research.

The open-source software tool `knsolver`[5] can be used to generate plots of solutions to the time-neutral and zero-time metrics. It uses numerical methods on the standard form of the Kerr-Newman[6] metric which is flexible but slow, especially for precision results. Converting a time-neutral metric to polynomial form may allow for fast exact solutions, depending on the degree of the polynomial.

The time-neutral metric should not be confused with metrics of neutral signature[7–11] $(+,+,-,-)$, which are often referred to as "neutral metrics" in mathematical literature.

POLYNOMIAL FORM

As shown in the proof below, the time-neutral exterior Schwarzschild metric in spherical coordinates (t, r, θ, φ), with test particle velocities converted to v^2/c^2, as a polynomial in r and total mass M are given by:

$$\left(\frac{v_r^2}{c^2} + \frac{v_\Omega^2}{c^2}\right) r_\mathrm{s} r^3 + \left(1 - \frac{v_\Omega^2}{c^2}\right) r_\mathrm{s}^2 r^2 - r_\mathrm{s}^3 r = 0, \tag{2}$$

$$\frac{8G^3 r}{c^6} M^3 - \left(1 - \frac{v_\Omega^2}{c^2}\right) \frac{4G^2 r^2}{c^4} M^2 - \left(\frac{v_r^2}{c^2} + \frac{v_\Omega^2}{c^2}\right) \frac{2Gr^3}{c^2} M = 0, \tag{3}$$

$$r_\mathrm{s} = \frac{2GM}{c^2}, \quad (r \geq R), \quad \left(v_r^2/c^2 \neq 0\right). \tag{4}$$

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* kevin@loch.me

[1] In metrics where angular momentum $J \neq 0$, an arbitrary finite non-zero value can be assigned to the remaining linear dt factors without affecting the neutral metric. `knsolver` sets this to t_P, but other values such as 1, or even -1 also work. The remaining linear $d\varphi$ factors must then be derived from $d\varphi = v_\varphi dt/r$.

As both are cubic with no constant term, the two non-zero roots of each can be recovered from the reduced quadratic equations:

$$\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^2+\left(1-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r-r_{\mathrm{s}}^3=0, \tag{5}$$

$$\frac{8G^3 r}{c^6} M^2-\left(1-\frac{v_\Omega^2}{c^2}\right) \frac{4G^2 r^2}{c^4} M-\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right) \frac{2G r^3}{c^2}=0, \tag{6}$$

$$r_{\mathrm{s}}=\frac{2GM}{c^2}, \quad (r \geq R), \quad \left(v_r^2/c^2 \neq 0\right). \tag{7}$$

This agrees with the two non-zero roots for each implied by plots generated from `knsolver`[5] when electric charge Q and angular momentum J are set to zero.

PROOF

Deriving the polynomials involves basic algebraic manipulation but special care must be taken to not lose the degrees of freedom represented by the inverse sum $\left(1-\frac{r_{\mathrm{s}}}{r}\right)^{-1}$ in the dr^2 term. This proof also uses some substitutions that are not strictly necessary here but can be used to help manage the proliferation of length factors in the higher order metrics[6, 12, 13].

We begin with the exterior $(r \geq R)$ Schwarzschild metric[2] in spherical coordinates (t, r, θ, φ) and metric signature $(+,-,-,-)$:

$$ds^2=c^2 d\tau^2=\left(1-\frac{r_{\mathrm{s}}}{r}\right) c^2 dt^2-\left(1-\frac{r_{\mathrm{s}}}{r}\right)^{-1} dr^2-r^2 d\Omega^2, \quad (r \geq R), \tag{8}$$

$$r_{\mathrm{s}}=\frac{2GM}{c^2}, \quad d\Omega^2=d\theta^2+\sin\theta d\varphi^2. \tag{9}$$

Divide the metric by c^2dt^2 and set equal to one to obtain the time-neutral metric, also convert test particle velocities to v^2/c^2. This is the equation we will convert to polynomial form:

$$\frac{d\tau^2}{dt^2}=\left(1-\frac{r_{\mathrm{s}}}{r}\right)-\frac{v_r^2}{c^2}\left(1-\frac{r_{\mathrm{s}}}{r}\right)^{-1}-\frac{v_\Omega^2}{c^2}=1, \quad (r \geq R), \tag{10}$$

$$\frac{v_r^2}{c^2}=\frac{dr^2}{c^2 dt^2}, \quad \frac{v_\Omega^2}{c^2}=\frac{r^2 d\Omega^2}{c^2 dt^2}. \tag{11}$$

Multiply the dt^2 and $d\Omega^2$ terms by $\left(1-\frac{r_{\mathrm{s}}}{r}\right)/\left(1-\frac{r_{\mathrm{s}}}{r}\right)$, then simplify the denominator by dividing both sides by r. At this point we must add the constraint $v_r^2/c^2dt^2 \neq 0$ as we are combining the dr^2 term inverse length factor with the other terms in the denominator.

$$(r \geq R), \quad \left(v_r^2/c^2 \neq 0\right), \tag{12}$$

$$\frac{\left(1-\frac{r_{\mathrm{s}}}{r}\right)\left(1-\frac{r_{\mathrm{s}}}{r}\right)-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\left(1-\frac{r_{\mathrm{s}}}{r}\right)}{1-\frac{r_{\mathrm{s}}}{r}}=1, \tag{13}$$

$$\tag{14}$$

$$\frac{\left(1-\frac{r_{\mathrm{s}}}{r}\right)\left(1-\frac{r_{\mathrm{s}}}{r}\right)-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\left(1-\frac{r_{\mathrm{s}}}{r}\right)}{r-r_{\mathrm{s}}}=\frac{1}{r}. \tag{15}$$

Multiply out the dt^2 term product, then simplify the numerator by multiplying both sides by r^2, then group by terms of r being especially careful with signs of combined factors:

$$\frac{1-\frac{2r_{\mathrm{s}}}{r}+\frac{r_{\mathrm{s}}^2}{r^2}-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\left(1-\frac{r_{\mathrm{s}}}{r}\right)}{r-r_{\mathrm{s}}}=\frac{1}{r}, \tag{16}$$

$$\frac{r^2 - 2r_\mathrm{s}r + r_\mathrm{s}^2 - \frac{v_r^2}{c^2}r^2 - \frac{v_\Omega^2}{c^2}\left(r^2 - r_\mathrm{s}r\right)}{r - r_\mathrm{s}} = r, \tag{17}$$

$$\frac{\left(1 - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\right)r^2 - \left(2 - \frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r + r_\mathrm{s}^2}{r - r_\mathrm{s}} = r. \tag{18}$$

At this point we introduce temporary variables X, Y, to help capture the extra degrees of freedom in the inverse sum. Let X be the numerator on the left side of equation 18:

$$\frac{X}{r - r_\mathrm{s}} = r, \tag{19}$$

$$X = \left(1 - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\right)r^2 - \left(2 - \frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r + r_\mathrm{s}^2. \tag{20}$$

The denominator sum is now split between numerators X and Y:

$$\frac{X}{r - r_\mathrm{s}} = \frac{X}{r} + \frac{Y}{r_\mathrm{s}}, \tag{21}$$

$$\frac{X}{r - r_\mathrm{s}} = \frac{Xr_\mathrm{s} + Yr}{r_\mathrm{s}r}. \tag{22}$$

It is now safe to multiply both sides by $(r - r_\mathrm{s})\,r_\mathrm{s}r$, then we group by terms of X and Y:

$$Xr_\mathrm{s}r = \left(Xr_\mathrm{s} + Yr\right)\left(r - r_\mathrm{s}\right), \tag{23}$$

$$Xr_\mathrm{s}r = Xr_\mathrm{s}r - Xr_\mathrm{s}^2 + Yr^2 - Yr_\mathrm{s}r, \tag{24}$$

$$Xr_\mathrm{s}^2 = Y\left(r^2 - r_\mathrm{s}r\right). \tag{25}$$

Substitute symbols X_f, Y_f, for length factors associated with X and Y, then solve for Y:

$$XX_f = YY_f, \tag{26}$$

$$X_f = r_\mathrm{s}^2, \qquad Y_f = r^2 - r_\mathrm{s}r, \tag{27}$$

$$Y = \frac{XX_f}{Y_f}. \tag{28}$$

With equations 19 and 22, multiply by $r_\mathrm{s}r$ and substitute for Y using equation 28:

$$\frac{Xr_\mathrm{s} + Yr}{r_\mathrm{s}r} = r, \tag{29}$$

$$Xr_\mathrm{s} + \frac{XX_fr}{Y_f} = r_\mathrm{s}r^2. \tag{30}$$

Multiply by Y_f, then group by terms of X:

$$XY_fr_\mathrm{s} + XX_fr = Y_fr_\mathrm{s}r^2, \tag{31}$$

$$X\left(Y_fr_\mathrm{s} + X_fr\right) - Y_fr_\mathrm{s}r^2 = 0. \tag{32}$$

Substitute symbols $F_1, F_{2,}, F_3$, for the combined length factors:

$$X\left(F_1+F_2\right)-F_3=0, \tag{33}$$

$$F_1=Y_f r_{\mathrm{s}}, \quad F_2=X_f r, \quad F_3=Y_f r_{\mathrm{s}} r^2. \tag{34}$$

With equations 27, and 34, solve F_1+F_2, and with equation 20, $X\left(F_1+F_2\right)$:

$$F_1=r_{\mathrm{s}} r^2-r_{\mathrm{s}}^2 r, \quad F_2=r_{\mathrm{s}}^2 r, \tag{35}$$

$$F_1+F_2=r_{\mathrm{s}} r^2, \tag{36}$$

$$X\left(F_1+F_2\right)=\left[\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right) r^2-\left(2-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r+r_{\mathrm{s}}^2\right] r_{\mathrm{s}} r^2, \tag{37}$$

$$X\left(F_1+F_2\right)=\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^4-\left(2-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^3+r_{\mathrm{s}}^3 r^2. \tag{38}$$

Solve F_3 and $X\left(F_1+F_2\right)-F_3$:

$$F_3=r_{\mathrm{s}} r^4-r_{\mathrm{s}}^2 r^3, \tag{39}$$

$$X\left(F_1+F_2\right)-F_3=\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^4-\left(2-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^3+r_{\mathrm{s}}^3 r^2-r_{\mathrm{s}} r^4+r_{\mathrm{s}}^2 r^3. \tag{40}$$

With equations 33 and 40, divide by r as the smallest degree of r is 2:

$$\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^4-\left(2-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^3+r_{\mathrm{s}}^3 r^2-r_{\mathrm{s}} r^4+r_{\mathrm{s}}^2 r^3=0, \tag{41}$$

$$\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^3-\left(2-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^2+r_{\mathrm{s}}^3 r-r_{\mathrm{s}} r^3+r_{\mathrm{s}}^2 r^2=0. \tag{42}$$

Convert to standard polynomial form in r by grouping terms by r, then inverting the sign of each term, and applying the constraints from equation 12:

$$\left(-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^3-\left(1-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^2+r_{\mathrm{s}}^3 r=0, \tag{43}$$

$$\boxed{\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^3+\left(1-\frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^2-r_{\mathrm{s}}^3 r=0, \quad \left(r \geq R, \quad v_r^2/c^2 \neq 0\right)} \tag{44}$$

Convert to standard polynomial form in M by substituting $r_s=\frac{2GM}{c^2}$, then grouping terms by M and inverting signs again:

$$\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right) \frac{2GM}{c^2} r^3+\left(1-\frac{v_\Omega^2}{c^2}\right) \frac{4G^2M^2}{c^4} r^2-\frac{8G^3M^3}{c^6} r=0, \tag{45}$$

$$\boxed{\frac{8G^3 r}{c^6} M^3-\left(1-\frac{v_\Omega^2}{c^2}\right) \frac{4G^2 r^2}{c^4} M^2-\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right) \frac{2Gr^3}{c^2} M=0, \quad \left(r \geq R, \quad v_r^2/c^2 \neq 0\right)} \tag{46}$$

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