Incomplete gamma function and Pi

Edgar Valdebenito

February 19, 2024

ABSTRACT: In this note we give three double series for Pi.

Keywords: Incomplete gamma function, Bernoulli numbers, number Pi, double series.

I. Introduction

The incomplete gamma function is defined by

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t}\, t^{\alpha-1}\, dt \quad , \ \mathbb{R}(\alpha) > 0 \tag{1}$$

Some properties:

$$\Gamma(z, 0) = \Gamma(z) \tag{2}$$

$$\Gamma(a, \infty) = 0 \tag{3}$$

$$\Gamma(n+1, x) = n!\, e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!} \quad , \ n = 0, 1, 2, 3, \ldots \tag{4}$$

$$\Gamma(\alpha, \ x\,y) = y^\alpha\, e^{-x\,y} \int_0^\infty e^{-t\,y} (t+x)^{\alpha-1}\, dt \ , \ \mathbb{R}(y) > 0, \ x > 0, \ \mathbb{R}(\alpha) > 1 \tag{5}$$

For details see references [1],[5].

II. Bernoulli numbers

The Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{(2\,n)!} x^{2\,n} \quad , \ |x| < 2\,\pi \tag{6}$$

$$\{B_n\,, \ n \geq 1\} = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots\right\} \tag{7}$$

$$B_n = (-1)^n \sum_{k=2}^{2\,n+1} \frac{(-1)^k}{k} \binom{2\,n+1}{k} \sum_{m=1}^{k-1} m^{2\,n} \quad , \ n = 1, 2, 3, \ldots \tag{8}$$

$$B_n = (-1)^{n+1} \sum_{k=0}^{2\,n} \frac{1}{k+1} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^{2\,n} \quad , \ n = 1, 2, 3, \ldots \tag{9}$$

For details see references [1],[5].

III. Generalized incomplete gamma function

The generalized incomplete gamma function is defined by

$$\Gamma(a, z_0, z_1) = \Gamma(a, z_0) - \Gamma(a, z_1) \tag{10}$$

IV. Three double series for Pi

$$\pi = 24\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}\left(2^{2n}-1\right)B_n}{(2n)!}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2^{-2k}}{(2k+1)^{2n-1}}\,\Gamma\!\left(2n,\ \frac{2k+1}{2}\ln\!\left(\frac{9+4\sqrt{3}}{7+4\sqrt{3}}\right),\ \frac{2k+1}{2}\ln\!\left(\frac{5+2\sqrt{2}}{3+2\sqrt{2}}\right)\right) \tag{11}$$

$$\pi = 12\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}\left(2^{2n}-1\right)B_n}{(2n)!}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2^{-2k}}{(2k+1)^{2n-1}}\,\Gamma\!\left(2n,\ \frac{2k+1}{2}\ln\!\left(\frac{9+4\sqrt{3}}{7+4\sqrt{3}}\right),\ \frac{2k+1}{2}\ln\!\left(\frac{5}{3}\right)\right) \tag{12}$$

$$\pi = 24\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}\left(2^{2n}-1\right)B_n}{(2n)!}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2^{-2k}}{(2k+1)^{2n-1}}\,\Gamma\!\left(2n,\ \frac{2k+1}{2}\ln\!\left(\frac{5+2\sqrt{2}}{3+2\sqrt{2}}\right),\ \frac{2k+1}{2}\ln\!\left(\frac{5}{3}\right)\right) \tag{13}$$

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \tag{14}$$

V. The numbers $C(n)$

Define

$$C(n+1) = \frac{1}{n+1}\sum_{k=0}^{n}(n-k+1)\,S(n-k+1)\,C(k) \quad,\ C(0) = 1\ ,\ n = 0, 1, 2, 3, \ldots \tag{15}$$

where

$$S(0) = 0,\ S(1) = \frac{1}{2},\ S(2n+1) = 0\ ,\ n = 1, 2, 3, \ldots \tag{16}$$

$$S(2n) = \frac{3\,(-1)^{n-1}\,2^{2n-1}\,B_n}{(2n)\,(2n)!}\ ,\ n = 1, 2, 3, \ldots \tag{17}$$

we have

$$\{C(n),\ n \geq 0\} = \left\{1,\ \frac{1}{2},\ -\frac{1}{8},\ -\frac{5}{48},\ \frac{7}{640},\ \frac{19}{1280},\ -\frac{869}{967680},\ -\frac{715}{387072},\ \ldots\right\} \tag{18}$$

VI. Alternative series

$$\pi = 3 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{2n} \left(2^{2n}-1\right) B_n}{(2n)!} \sum_{k=0}^{\infty} \frac{C(k)}{2n+k-(3/2)} \left(\operatorname{arctanh}\left(\frac{1}{4}\right)\right)^{2n+k-(3/2)} \tag{19}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{2n} \left(2^{2n}-1\right) B_n}{(2n)!} \sum_{k=0}^{\infty} \frac{C(k)}{2n+k-(3/2)} \left(\operatorname{arctanh}\left(\frac{1}{4+2\sqrt{2}}\right)\right)^{2n+k-(3/2)} \tag{20}$$

$$\pi = 6 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{2n} \left(2^{2n}-1\right) B_n}{(2n)!} \sum_{k=0}^{\infty} \frac{C(k)}{2n+k-(3/2)} \left(\operatorname{arctanh}\left(\frac{1}{8+4\sqrt{3}}\right)\right)^{2n+k-(3/2)} \tag{21}$$

Define

$$D(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\, 2^{2k+2} \left(2^{2k+2}-1\right) B_{k+1}}{(2k+2)!} C(n-2k) \quad , \; n = 0, 1, 2, 3, \ldots \tag{22}$$

$$\{D(n),\, n \geq 0\} = \left\{1,\, \frac{1}{2},\, -\frac{11}{24},\, -\frac{13}{48},\, \frac{119}{640},\, \frac{1339}{11\,520},\, -\frac{72\,749}{967\,680},\, -\frac{18\,451}{387\,072},\, \ldots\right\} \tag{23}$$

we have

$$\pi = 3 \sum_{n=0}^{\infty} \frac{D(n)}{n+(1/2)} \left(\operatorname{arctanh}\left(\frac{1}{4}\right)\right)^{n+(1/2)} \tag{24}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{D(n)}{n+(1/2)} \left(\operatorname{arctanh}\left(\frac{1}{4+2\sqrt{2}}\right)\right)^{n+(1/2)} \tag{25}$$

$$\pi = 6 \sum_{n=0}^{\infty} \frac{D(n)}{n+(1/2)} \left(\operatorname{arctanh}\left(\frac{1}{8+4\sqrt{3}}\right)\right)^{n+(1/2)} \tag{26}$$

VII. References

1. Abramowitz, M., and Stegun, I.A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematical Series 55, National Bureau of Standards, Washington, DC, Dover, New York, 1965.
2. Andrews, G.E., Askey, R., and Roy, R., Special Functions. Encyclopedia of Mathematics and its Applications, 71, 1999, Cambridge University Press.
3. Artin, E., The Gamma Function. Holt, Rinehart and Winston, New York, 1964.
4. Beckmann, P., A History of π. 2nd ed., Golem Press, Boulder, CO, 1971.
5. Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series and Products. 5th ed., ed. Alan Jeffrey. Academic Press, 1994.