Diophantine Nth-tuples

a record of Diophantine equations of sums of terms of various degrees

theorem 1: For $p,q \in \mathbb{N}$:

$$(p^3+q^3)^3 = (p^3-q^3)^3 + 6(p^2q)^3$$

proof:

$$(p^3+q^3)^3 = p^9 + 3p^6(q^3) + 3p^3(q^3)^2 + (q^3)^3$$
$$(p^3-q^3)^3 = p^9 - 3p^6(q^3) + 3p^3(q^3)^2 - (q^3)^3$$
$$\Rightarrow (p^3+q^3)^3 - (p^3-q^3)^3 = 6p^6q^3 = 6(p^2q)^3 + 2(q^3)^3$$
$$\Rightarrow (p^3+q^3)^3 = (p^3-q^3)^3 + 6(p^2q)^3 + 2(q^3)^3$$

□

theorem 2: For $a,b,p,q,m \in \mathbb{N}$:

$$(ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^h q^{(m-h)})^m$$

proof:

$$(ap^m+bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} (ap^m)^h (bq^m)^{(m-h)}$$
$$(ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} (ap^m)^h (-bq^m)^{(m-h)}$$
$$\Rightarrow (ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} \left[(ap^m)^h (bq^m)^{(m-h)} - (ap^m)^h (-bq^m)^{(m-h)}\right]$$
$$\Rightarrow (ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} \left[a^h (p^m)^h b^{(m-h)} (q^m)^{(m-h)} - a^h (p^m)^h (-b)^{(m-h)} (q^m)^{(m-h)}\right]$$
$$\Rightarrow (ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^m)^h (q^m)^{(m-h)}$$
$$\Rightarrow (ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^m)^h (q^m)^{(m-h)}$$
$$\Rightarrow (ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^h)^m (q^{(m-h)})^m$$
$$\Rightarrow (ap^m+bq^m)^m - (ap^m-bq^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^h q^{(m-h)})^m$$

□

theorem 3: For $a,b,p,q,m,n \in \mathbb{N}$:

$$(ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^h q^{(m-h)})^n$$

proof:

$$(ap^n+bq^n)^m = a^m p^{mn} + \sum_{h=0}^{m} \binom{m}{h} (ap^n)^h (bq^n)^{(m-h)}$$
$$(ap^n-bq^n)^m = a^m p^{mn} + \sum_{h=0}^{m} \binom{m}{h} (ap^n)^h (-bq^n)^{m-}$$
$$\Rightarrow (ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} \left[(ap^n)^h (bq^n)^{(m-h)} - (ap^n)^h (-bq^n)^{(m-h)}\right]$$
$$\Rightarrow (ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} \left[a^h (p^n)^h b^{(m-h)} (q^n)^{(m-h)} - a^h (p^n)^h (-b)^{(m-h)} (q^n)^{(m-h)}\right]$$
$$\Rightarrow (ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^n)^h (q^n)^{(m-h)}$$
$$\Rightarrow (ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^n)^h (q^n)^{(m-h)}$$
$$\Rightarrow (ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^h)^n (q^{(m-h)})^n$$
$$\Rightarrow (ap^n+bq^n)^m - (ap^n-bq^n)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{(m-h)} \left[1-(-1)^{(m-h)}\right] (p^h q^{(m-h)})^n$$

□

examples:

$$(ap^n+bq^n)^3 - (ap^n-bq^n)^3 = \sum_{h=0}^{3} \binom{3}{h} a^h b^{(3-h)} \left[1-(-1)^{(3-h)}\right] (p^h q^{(3-h)})^n$$
$$= 2(q^3)^n + 6a^2b(p^2q)^n$$

$a = 6, b = 1; n = 3$:

$$(6p^3+q^3)^3 - (6p^3-q^3)^3 = 2(q^3)^3 + 6^3(p^2q)^3$$

$p = 7, q = 5; n = 3$:

$$(6 \cdot 7^3 + 5^3)^3 - (6 \cdot 7^3 - 5^3)^3 = (6 \cdot 7^2 \cdot 5)^3 + 2 \cdot 125^3$$

$$(2183)^3 - (1933)^3 = (1470)^3 + 3,906,250$$
$$10,403,062,487 - 7,222,633,237 = 3,180,429,250 = 3,176,523,000 + 3,906,250 \quad ✓$$

$$\begin{aligned}(ap^n + bq^n)^4 - (ap^n - bq^n)^4 &= \sum_{h=0}^{4} \binom{4}{h} a^h b^{(4-h)} \left[1 - (-1)^{(4-h)}\right] (p^h q^{(4-h)})^n \\ &= 8ab^3(pq^3)^n + 8a^3b(p^3q)^n \\ &= 8ab[b^2(pq^3)^n + 8a^2(p^3q)^n]\end{aligned}$$

$a = 2, b = 1; n = 4$:

$$\begin{aligned}(8p^4 + q^4)^4 - (8p^4 - q^4)^4 &= 8 \bullet 2(pq^3)^4 + 8 \bullet 8(2^4 3^4)(p^3q)^4 \\ &= (2pq^3)^4 + 64(p^3q)^4\end{aligned}$$

$$\begin{aligned}(ap^n + bq^n)^5 - (ap^n - bq^n)^5 &= \sum_{h=0}^{5} \binom{5}{h} a^h b^{(5-h)} \left[1 - (-1)^{(5-h)}\right] (p^h q^{(5-h)})^n \\ &= 20a^2b^3(p^2q^3)^n + 10a^4b(p^4q^1)^n\end{aligned}$$

$a = 10, b = 1; n = 5$:

$$(10p^5 + q^5)^5 - (10p^5 - q^5)^5 = 2000(p^2q^3)^5 + (10p^4q^1)^5$$

theorem 4: For $a_j, b_j, p_j, q_j, m, j \in \mathbb{N}$:

$$(a_j p_j^m + b_j q_j^m)^m - (a_j p_j^m - b_j q_j^m)^m = \sum_{h=0}^{m} \binom{m}{h} a_j^h b_j^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \left(p_j^h q_j^{(m-h)}\right)^m$$

$$\Rightarrow \sum_{j=1}^{N} [(a_j p_j^m + b_j q_j^m)^m - (a_j p_j^m - b_j q_j^m)^m] = \sum_{h=0}^{m} \binom{m}{h} \left[1 - (-1)^{(m-h)}\right] \sum_{j=1}^{N} \left[a_j^h b_j^{1(m-h)} \left(p_j^h q_j^{(m-h)}\right)^m\right]$$

proof:

$$(a_j p_j^m + b_j q_j^m)^m - (a_j p_j^m - b_j q_j^m)^m = \sum_{h=0}^{m} \binom{m}{h} a_j^h b_j^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \left(p_j^h q_j^{(m-h)}\right)^m$$

$$\Rightarrow \sum_{j=1}^{N} [(a_j p_j^m + b_j q_j^m)^m - (a_j p_j^m - b_j q_j^m)^m] = \sum_{j=1}^{N} \left[\sum_{h=0}^{m} \binom{m}{h} a_j^h b_j^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \left(p_j^h q_j^{(m-h)}\right)^m\right]$$

$$\Rightarrow \sum_{j=1}^{N} [(a_j p_j^m + b_j q_j^m)^m - (a_j p_j^m - b_j q_j^m)^m] = \sum_{h=0}^{m} \binom{m}{h} \left[1 - (-1)^{(m-h)}\right] \sum_{j=1}^{N} \left[a_j^h b_j^{1(m-h)} \left(p_j^h q_j^{(m-h)}\right)^m\right]$$

□

corollary 4.1: For $a_j, b_j, p, q, m, j \in \mathbb{N}$:

$$[(a_j p^m + b_j q^m)^m - (a_j p^m - b_j q^m)^m] = \sum_{h=0}^{m} \binom{m}{h} a_j^h b_j^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] (p^h q^{(m-h)})^m$$

$$\Rightarrow \sum_{j=1}^{N} [(a_j p^m + b_j q^m)^m - (a_j p^m - b_j q^m)^m] = \sum_{h=0}^{m} \left(\sum_{j=1}^{N} \left[a_j^h b_j^{1(m-h)}\right]\right) \binom{m}{h} \left[1 - (-1)^{(m-h)}\right] (p^h q^{(m-h)})^m$$

proof:

$$(a_j p^m + b_j q^m)^m - (a_j p^m - b_j q^m)^m = \sum_{h=0}^{m} \binom{m}{h} a_j^h b_j^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] (p^h q^{(m-h)})^m$$

$$\Rightarrow \sum_{j=1}^{N} [(a_j p^m + b_j q^m)^m - (a_j p^m - b_j q^m)^m] = \sum_{j=1}^{N} \left[\sum_{h=0}^{m} \binom{m}{h} a_j^h b_j^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] (p^h q^{(m-h)})^m\right]$$

$$\Rightarrow \sum_{j=1}^{N} [(a_j p^m + b_j q^m)^m - (a_j p^m - b_j q^m)^m] = \sum_{h=0}^{m} \left(\sum_{j=1}^{N} \left[a_j^h b_j^{1(m-h)}\right]\right) \binom{m}{h} \left[1 - (-1)^{(m-h)}\right] (p^h q^{(m-h)})^m$$

□

similarly:

corollary 4.2: For $a, b, p_j, q_j, m, j \in \mathbb{N}$:

$$(ap_j^m + bq_j^m)^m - (ap_j^m - bq_j^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \left(p_l^h q_j^{(m-h)}\right)^m$$

$$\Rightarrow \sum_{j=1}^{N} [(ap_j^m + bq_j^m)^m - (ap_j^m - bq_j^m)^m] = \sum_{h=0}^{m} \binom{m}{h} a^h b^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \sum_{j=1}^{N} \left(p_l^h q_j^{(m-h)}\right)^m$$

proof:

$$(ap_j^m + bq_j^m)^m - (ap_j^m - bq_j^m)^m = \sum_{h=0}^{m} \binom{m}{h} a^h b^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \left(p_l^h q_j^{(m-h)}\right)^m$$

$$\Rightarrow \sum_{j=1}^{N} [(ap_j^m + bq_j^m)^m - (ap_j^m - bq_j^m)^m] = \sum_{j=1}^{N} \left[\sum_{h=0}^{m} \binom{m}{h} a^h b^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \left(p_l^h q_j^{(m-h)}\right)^m\right]$$

$$\Rightarrow \sum_{j=1}^{N} [(ap_j^m + bq_j^m)^m - (ap_j^m - bq_j^m)^m] = \sum_{h=0}^{m} \binom{m}{h} a^h b^{1(m-h)} \left[1 - (-1)^{(m-h)}\right] \sum_{j=1}^{N} \left(p_l^h q_j^{(m-h)}\right)^m$$

□