

Alpha is a Wavenumber and the Uncertainty Principle Exists

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Abstract

A simple derivation of alpha, the fine structure constant, using Coulomb's law and the Planck-Einstein relations. I argue that alpha represents the minimum uncertainty between wavenumber and radial distance. This is like the uncertainty between momentum (wavelength) and position. The fine structure constant is related to this uncertainty principle but in spherical coordinates using wavenumber for momentum and radius for distance. Wavenumber is defined as the inverse of the wavelength per unit distance. This is equivalent to saying that alpha is about 137 wavelengths per unit distance of radius. I go on to show this provides the correct ionization wavelengths for the hydrogen atom. Using whole integers, n number of energy levels, allowed me to derive the Rydberg formula. Alpha is nearly an integer number because we are using a wavenumber. It is not a mystery to find integer values when wavenumbers are used. This derivation is equivalent to that of the Bohr model but without needing to use classical ideas of electrons in orbit around the nucleus like planets in orbit around the sun.

1 Deriving α

The force between a single proton and electron that each have a charge e is

$$F = \frac{e^2}{4\pi\epsilon_0 r^2} \quad (1)$$

The integral of the force is the potential energy between the two charges

$$U = - \int F dr \quad (2)$$

$$U = - \int \frac{e^2}{4\pi\epsilon_0 r^2} dr \quad (3)$$

$$U = \frac{e^2}{4\pi\epsilon_0 r} \quad (4)$$

The photon is the force carrying particle of the electromagnetic force. The momentum and energy of a photon is a function of the wavelength λ and the frequency ν

$$P = \frac{h}{\lambda} \quad (5)$$

$$E = h\nu \quad (6)$$

The angular form with $k = \frac{2\pi}{\lambda}$, $\omega = 2\pi\nu$, and $\hbar = \frac{h}{2\pi}$ is

$$P = \hbar k \quad (7)$$

$$E = \hbar\omega \quad (8)$$

The velocity of the photon is a constant, the speed of light, and is defined as

$$\lambda\nu = c \quad (9)$$

The energy, $E = \hbar k$, can be written using the wavenumber

$$E = \hbar ck \quad (10)$$

When the electron drops to a lower state it emits a photon and the electron's potential energy decreases. When the electron absorbs a photon it jumps to a higher excited state and the potential energy of the electron increases. If the potential energy is zero in the ground state then the potential energy required for the electron to leave the proton is $U = \hbar ck$.

$$\hbar ck = \frac{e^2}{4\pi\epsilon_0 r} \quad (11)$$

Solving for k

$$k = \frac{e^2}{4\pi\epsilon_0\hbar cr} \quad (12)$$

α is defined as

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \quad (13)$$

Therefore the wavenumber of the photon between two charges is

$$k = \frac{\alpha}{r} \quad (14)$$

2 Deriving the Hydrogen Ionization Wavelength Using the Generalized Uncertainty Principle

The equation $k = \frac{\alpha}{r}$ can be rewritten as $2\pi r = \alpha\lambda$. The ionization energy, the energy required to remove the electron from the ground state of the hydrogen atom, is 13.6eV. The Bohr radius, the distance from the nucleus to the ground state, is 5.292×10^{-8} m. Setting r to the Bohr radius and solving for λ gives a wavelength of 4.555×10^{-11} m.

$$2\pi r = \alpha\lambda = \frac{2\pi(5.291 \times 10^{-11} \text{m})}{\frac{1}{137}} = 4.555 \times 10^{-8} \text{m}$$

The ionization energy of hydrogen, the energy to remove the electron from the ground state, requires a photon with a wavelength of 9.11×10^{-8} m. The equation derived above gives the wrong answer for the photon's wavelength. However, the correct ionization wavelength is found by multiplying $\lambda = 4.555 \times 10^{-11}$ by 2. The factor of 2 is a consequence of the uncertainty principle. The uncertainty principle provides the relationship between position and momentum as

$$\Delta X \Delta P \geq \frac{\hbar}{2} \quad (15)$$

For

$$kr = \alpha \quad (16)$$

Set $\Delta k = \Delta P$ and $\Delta r = \Delta X$

$$\Delta k \Delta r \geq \frac{\alpha}{2} \quad (17)$$

The generalized uncertainty principle for two observables, a and b ,

$$\sigma_a \sigma_b \geq \frac{ab - ba}{2} \quad (18)$$

provides the missing factor of two and suggests the radial distance r and wavenumber k has an uncertainty related to α

$$[r, k] = rk - kr = \alpha \quad (19)$$

The correct equation is

$$4\pi r \geq \alpha \lambda \quad (20)$$

The correct wavelength of a photon that ionizes hydrogen with an electron in the ground state at a radius of a_o

$$\lambda \geq \frac{4\pi a_o}{\frac{1}{137}} \geq 9.1103 \times 10^8 \text{ m}$$

3 Deriving The Rydberg Formula For Hydrogen And Hydrogen Like Atoms

Discrete steps in the change of wavelength and radius are needed to quantize the equation. Increasing Δr by n quantizes the radius and increasing Δk by n quantizes the wavenumber. The change in radius and wavenumber only occurs in discrete whole numbers of quanta.

$$\Delta k \Delta r \geq \frac{\alpha}{2} \quad (21)$$

$$(\Delta n_r) r (\Delta n_k) k \geq \frac{\alpha}{2} \quad (22)$$

Setting k to $\frac{2\pi}{\lambda}$ is the quantized equation to begin describing the change wavelength of the photon when transitioning from one state to another state.

$$(\Delta n_r \Delta n_k) r \frac{2\pi}{\lambda} \geq \frac{\alpha}{2} \quad (23)$$

$$4\pi (\Delta n_r \Delta n_k) r \geq \alpha \lambda \quad (24)$$

Substituting in the Bohr radius, a_o .

$$4\pi \Delta n_r \Delta n_k a_o \geq \alpha \lambda \quad (25)$$

The Δn values represent the standard deviation, $\Delta n = \sqrt{n_f^2 - n_i^2}$. The uncertainty principle restricts the possible n_r and n_k integers. As Δn_r gets larger Δn_k can get smaller. As Δn_k gets larger Δn_r can get smaller. This is true as long as the product of $(\Delta n_k \Delta n_r) r k$ is greater than or equal to $\frac{\alpha}{2}$ and as long as $n_r n_k > 0$. The quantized values can never be zero or that would violate the uncertainty principle. Therefore $n_r, n_k \geq 1$. The lowest possible state, the ground state, is where $n_r, n_k = 1$. When the states of $\Delta n_r \Delta n_k$ are equal then their product is

$$\Delta n_r \Delta n_k = \sqrt{n_f^2 - n_i^2} \sqrt{n_f^2 - n_i^2} = n_f^2 - n_i^2 \quad (26)$$

$$\Delta n_r \Delta n_k = n_f^2 - n_i^2 \quad (27)$$

Substituting in $\Delta n_r \Delta n_k = n_f^2 - n_i^2$

$$4\pi a_o (n_f^2 - n_i^2) \geq \alpha \lambda \quad (28)$$

$$\frac{1}{\lambda} \geq \frac{\alpha}{4\pi a_o} \frac{1}{n_f^2 - n_i^2} \quad (29)$$

Using Rydberg's constant, $R = \frac{\alpha}{4\pi a_o}$, completes the Rydberg formula derivation

$$\boxed{\frac{1}{\lambda} \geq \frac{R}{n_f^2 - n_i^2}} \quad (30)$$

The equation describing hydrogen like atoms, atoms with multiple protons but only a single electron, is derived in a similar way. The number of protons Z of charge e and a single electron of charge e require a potential energy to ionize the atom of

$$\hbar ck = \frac{Ze^2}{4\pi\epsilon_o r} \quad (31)$$

$$k = \frac{Z\alpha}{r} \quad (32)$$

$$\frac{2\pi r}{Z} = \alpha \lambda \quad (33)$$

The uncertainty principle cannot restricts the product of $(\Delta n_r)r(\Delta n_k)k$ to be greater than or equal to $\frac{\alpha}{2}$. Therefore to not violate the uncertainty principle dividing one side by Z must mean the other side has to be multiplied by Z .

$$\frac{\Delta r \Delta k}{Z} = \frac{(n_r n_k)rk}{Z} \geq \frac{Z\alpha}{2} \quad (34)$$

Writing $k = \frac{2\pi}{\lambda}$ and setting r to the Bohr radius, a_o gives

$$4\pi a_o n_r n_k \geq Z^2 \alpha \lambda \quad (35)$$

$$\frac{1}{\lambda} \geq RZ^2 \frac{1}{n_r n_k} \quad (36)$$

Using $n_r n_k = n_f^2 - n_i^2$ produces the Rydberg formula for hydrogen like atoms

$$\boxed{\frac{1}{\lambda} \geq RZ^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)} \quad (37)$$