EULER–MASCHERONI CONSTANT IS IRRRATIONAL

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Abstract. The proof is written.
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It is not known if this constant is irrational, let alone transcendental (Wells 1986, p. 28). The famous English mathematician G. H. Hardy is alleged to have offered to give up his Savilian Chair at Oxford to anyone who proved gamma to be irrational (Havil 2003, p. 52), although no written reference for this quote seems to be known. Hilbert mentioned the irrationality of gamma as an unsolved problem that seems "unapproachable" and in front of which mathematicians stand helpless (Havil 2003, p. 97). Conway and Guy (1996) are "prepared to bet that it is transcendental," although they do not expect a proof to be achieved within their lifetimes. If gamma is a simple fraction a/b, then it is known that $b > 10^{10000}$ (Brent 1977; Wells 1986, p. 28), which was subsequently improved by T. Papanikolaou to $b > 10^{242080}$ (Havil 2003, p. 97).

Let a be any $\mathbb{Q}$ number. It is known that for any a the $\exp(a) \in \mathbb{I}$: see Ref. [2]. As $\exp(\ln a) = a$, either $F = \ln(a) \in \mathbb{I}$ or $F = \ln(a) \in \mathbb{Q}$. Latter is not possible because $\exp(F)$ would become an $\mathbb{I}$ number due to Ref. [2], but $\exp(F) = \exp(\ln(a)) = a \in \mathbb{Q}$ by direct evaluation.

Let K be an integer in the following.

$$O(K) = \gamma + \ln(K) - H_K \,,$$

where H_K is K-th harmonic number, $\gamma \approx 0.5772$ is E.-M. constant. In the limit $O(K \to \infty) = 0$. It is easy to show that $\hat{O}(K) = K\,O(K) = K\,\gamma + K\,\ln(K) - K\,H_K = 0$ in that limit too. First of all, the rate $\hat{O}(K+1) - \hat{O}(K) = O(K) + \ln(1 + 1/K) = 0$ in the limit. Secondly, a function

$$\gamma(k) = H_k - \ln(k) \,,$$

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$\hat{O}(K,k) = K\,\gamma(k) + K\,\ln(K) - K\,H_K = 0$ at $K = k$ with $k \to \infty$. These facts point to $\hat{O}(K) = 0$ at $K \to \infty$.

The integer part of $L = K\,H_K$ is greater than K. Therefore, the information about K, while K goes to infinity, is being written into the integer part of L; namely, every value of K corresponds to a certain value of the integer part of L. Hereby, H_K remains rational. Therefore, L does not turn into an irrational number.

So, $K\,\gamma + K\,\ln(K) - L = 0$ means $K\,\gamma + \mathbb{I} - \mathbb{Q} = 0$.

So, $\gamma \in \mathbb{I}$.

References

[1] Eric W. Weisstein, "Euler-Mascheroni Constant." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Euler-MascheroniConstant.html; D. Wells, The Penguin Dictionary of Curious and Interesting Numbers. Middlesex, England: Penguin Books, p. 28, 1986; J. Havil, Gamma: Exploring Euler's Constant. Princeton, NJ: Princeton University Press, 2003; J. H. Conway, and R. K. Guy, "The Euler-Mascheroni Number." In The Book of Numbers. New York: Springer-Verlag, pp. 260–261, 1996; R. P. Brent, "Computation of the Regular Continued Fraction for Euler's Constant." Math. Comput. 31, 771–777, 1977.

[2] Martin Aigner, Günter M. Ziegler. Proofs from THE BOOK (4th ed.). Berlin, New York: Springer-Verlag. pp. 27–36 (1998). doi:10.1007/978-3-642-00856-6.