PROOF OF RIEMANN HYPOTHESIS VIA ROBIN'S THEOREM

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Abstract. I show that the minimum of the function $F = \mathrm{e}^{\gamma} \ln(\ln n) - \sigma(n)/n$ is found to be positive. Therefore, $F > 0$ holds for any $n > 5040$.

MSC Class: 11M26, 11M06.

Robin's Theorem [1] states that if

$$F = \mathrm{e}^{\gamma} \ln(\ln n) - \frac{\sigma(n)}{n} > 0 \tag{1}$$

for $n > 5040$, where $\gamma \approx 0.577$ is the Euler–Mascheroni constant and $\sigma(n)$ is the sum-of-divisors function, the Riemann hypothesis is true.

In the following I use methods of functional analysis to show that the minimum of F has to be positive. According to the fundamental theorem of arithmetic, one has

$$n = \prod_{i=1}^{\kappa} p_i^{\alpha_i} \,. \tag{2}$$

Hence, F is a unique function of the powers α_g via $n = n_0\, p_g^{\alpha_g}$, where p_g is not a divisor of n_0. Methodologically, derivatives with respect to α_g can be taken by calculating finite differences [2]. Therefore,

$$\frac{\Delta p_g^{\alpha_g}}{\Delta \alpha_g} = p_g^{\alpha_g} - p_g^{\alpha_g - 1} = p_g^{\alpha_g}\,(1 - 1/p_g) \tag{3}$$

and, accordingly,

$$S(n) = \frac{\Delta}{\Delta \alpha_g} \ln(\ln n) = \frac{1}{n \ln n} \frac{\Delta n}{\Delta \alpha_g} = \frac{1 - 1/p_g}{\ln n} \,. \tag{4}$$

On the other hand,

$$\frac{\sigma(n)}{n} = A\, u(\alpha_g)\,, \qquad u(\alpha_g) = 1 + \frac{1}{p_g} + \frac{1}{p_g^2} + \ldots \frac{1}{p_g^{\alpha_g}} \,. \tag{5}$$

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p_g is not a divisor of A, as it is made obvious by a simple example in the Appendix. Again, the finite difference results in

$$\frac{\Delta}{\Delta\alpha_g}\left(\frac{\sigma(n)}{n}\right) = A\left(u(\alpha_g) - u(\alpha_g - 1)\right) = \frac{A}{p_g^{\alpha_g}}\,. \tag{6}$$

By analysis, the minimum of F can be found via

$$0 = \frac{\Delta F}{\Delta\alpha_g} = \mathrm{e}^\gamma\,\frac{1 - 1/p_g}{\ln\, n} - \frac{A}{p_g^{\alpha_g}} = \mathrm{e}^\gamma\,\frac{1 - 1/p_g}{\ln\, n} - \frac{\sigma(n)}{n\,u(\alpha_g)\,p_g^{\alpha_g}} = 0\,. \tag{7}$$

This means

$$\mathrm{e}^\gamma\,(1 - 1/p_g)\,(1 + p_g + p_g^2 + ... + p_g^{\alpha_g}) = \sigma(n)\,\frac{\ln\, n}{n}\,, \tag{8}$$

and resummed

$$\mathrm{e}^\gamma\,(1 - 1/p_g)\,\frac{p_g^{\alpha_g+1} - 1}{p_g - 1} = \mathrm{e}^\gamma\left(p_g^{\alpha_g} - \frac{1}{p_g}\right) = \sigma(n)\,\frac{\ln\, n}{n} \tag{9}$$

or

$$p_g^{\alpha_g} = \mathrm{e}^{-\gamma}\,\sigma(n)\,\frac{\ln\, n}{n} + \frac{1}{p_g}\,. \tag{10}$$

For a given n, this formula provides values for p_g and α_g. The higher p_g is chosen, the lower α_g is found. The highest prime $g = \kappa$ has $\alpha_\kappa = 1$. From Eq. (8), one has

$$\frac{\sigma(n)}{n} = \mathrm{e}^\gamma\,\frac{p_\kappa^2 - 1}{p_\kappa\,\ln\, n}\,, \tag{11}$$

and inserting to Eq. (1) one obtains

$$F\,p_\kappa\,\ln\, n = \mathrm{e}^\gamma\,p_\kappa\,\ln\, n\,\ln(\ln\, n) - \mathrm{e}^\gamma\,(p_\kappa^2 - 1)\,, \tag{12}$$

where n is a potential counter-example. As the powers in Eq. (2) up to $i = \kappa$ are at least 1, $\alpha_i \geq 1$, Eq. (10) gives

$$n \geq \prod_{i=1}^{\kappa} p_i = \exp(\theta(p_\kappa))\,, \tag{13}$$

where θ is the first Chebyshev function with $\lim_{x\to\infty}(\theta(x)/x) = 1$ [3]. Therefore, $n > \exp(p_\kappa/2)$ inequality, and the right hand of the Eq. (12) is positive. This means, that F cannot have negative values.

Appendix: on the sum-of-divisors function

As an example, I consider $n = 28 = 2 \cdot 14 = 2 \cdot 2 \cdot 7 = 2^2 \cdot 7$. Therefore, $\sigma(28) = (1+7)(1+2+2^2) = 1+2+4+7+14+28$ and

$$\frac{\sigma(28)}{28} = \frac{1+7}{7} \cdot \frac{1+2+2^2}{2^2}\,. \tag{A1}$$

Selecting $p_g = 2$, one has $A = (1+7)/7$ and $u_g = (1+2+2^2)/2^2$.

References

[1] Guy Robin, "Grandes valeurs de la fonction somme des diviseurs et hypothése de Riemann." J. Math. pures appl, 63(2): 187–213 (1984).
[2] Jordán, op. cit., p. 1 and Milne-Thomson, p. xxi. Milne-Thomson, Louis Melville (2000): The Calculus of Finite Differences (Chelsea Pub Co, 2000)
[3] Apostol, Tom M. (2010). Introduction to Analytic Number Theory. Springer.