THE SIGNATURE OF ABC CONJECTURE IS PROVEN

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ABSTRACT. Equivalent view of abc conjecture is proven. Some crucial properties of the abc conjecture are presented and proven. For example, there exist three numbers (a, b, c) that satisfy the abc conjecture for an arbitrary value c.
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1. INTRODUCTION

The abc conjecture states the following. For every positive real number ϵ, and a triplet (a, b, c) of pairwise coprime positive integers with $a+b = c$, one has $k \leq K(\epsilon)$, with $k = k(a, b; \epsilon) = c/r^{1+\epsilon} = (a+b)/r^{1+\epsilon}$, where $r = \text{rad}(a\,b\,c)$ is the radical of product of the three integers. By taking the maximum of $k(a, b)$ within the spectrum of pairwise coprime positive integers a and b, I obtain the function $K(\epsilon) = \max_{a,b} k(a, b)$. Using this function, the abc conjecture states that $K(\epsilon) \neq \infty$ for any $\epsilon \neq 0$. The conjecture is regarded as being unproven [1].

Excitedly, I began to think about all the possible applications of the ABC conjecture. I knew that if it were true, it would have far-reaching implications for many different fields. Here are just a few examples that I thought of: [2]

Firstly, the ABC conjecture would have a significant impact on cryptography. It would allow for more secure encryption methods to be developed, making it more difficult for hackers to break into sensitive information.

Secondly, the conjecture would also have implications for number theory. It could lead to new discoveries about prime numbers, which are the building blocks of all other numbers. This could potentially revolutionize the field of mathematics as a whole.

Finally, the ABC conjecture could also have practical applications in physics. It could help scientists to better understand the properties of matter and energy, and could lead to new breakthroughs in areas such as quantum mechanics.

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2. The signature

The abc conjecture implies that in the limit $c \to \infty$, one has $r = \infty$. Otherwise, for every single $\epsilon > 0$ one has $K(\epsilon) = \infty$. For arbitrary $m > 0$ one has

$$c/r^{1+m} = U\,W\,, \tag{1}$$

where

$$U = c/r^{1+\epsilon}\,, \quad W = r^{\epsilon}/r^{m}\,, \tag{2}$$

and $\epsilon > 0$ is arbitrary. For $\epsilon > m$, in the limit $r \to \infty$ the abc conjecture implies $U = 0$, as $W = \infty$; because the abc conjecture implies finiteness of $c/r^{1+m} < \infty$ as well. One concludes that in the limit $r \to \infty$, the abc conjecture implies $k = c/r^{1+\epsilon} = 0$. If, for some triplet, the $U \neq 0$ happens in the limit $r \to \infty$, the abc conjecture is wrong because then $c/r^{1+m} = \infty$. Therefore, the limit exists. Accordingly, in this limit there is an infinite number of triplets (a, b, c) with k arbitrarily close to zero. In other words, the abc conjecture implies that for an arbitrary constant $\delta > 0$ there is an infinite number of triplets (a, b, c) satisfying $c/r^{1+\epsilon} < \delta$, $c < \delta\, r^{1+\epsilon}$.

3. Realization of the signature

Because a, b, c have no common factors, one has $r = \mathrm{rad}(a\,b)\,\mathrm{rad}(c)$.

Accordingly, $c < \delta\,(\mathrm{rad}(ab))^{1+\epsilon}\,(\mathrm{rad}(c))^{1+\epsilon}$. Here and in the following, δ is a fixed parameter. Let us study such numbers c which have $c = \mathrm{rad}(c)$. Therefore, $1 < \delta\,(\mathrm{rad}(ab))^{1+\epsilon}\,(\mathrm{rad}(c))^{\epsilon}$. By increasing c, $\mathrm{rad}(c)$ tends to infinity, $(\mathrm{rad}(ab))^{1+\epsilon} \geq 1$, and there is an infinite amount of different primes. Therefore, the infinite amount of triplets satisfies $1 < \delta\,(\mathrm{rad}(ab))^{1+\epsilon}\,(\mathrm{rad}(c))^{\epsilon}$. This holds for any combination of a and b for a given $c = a + b$.

In the following, c is again an arbitrary integer. Because there are several ways to put $c = a + b$, k can take several values for a given c. The maximum value $\kappa(c) = \max k(c)$ saturates at zero at least for the $c = \mathrm{rad}\, c$, $c \to \infty$ conditions. Hereby, the radical is $R = (c/\kappa(c))^{1/(1+\epsilon)}$.

4. The case $a = \mathrm{rad}\, a$

Any number c (odd or even) can be presented as a sum $a + b = c$, where b is an (prime or non-prime) integer in the interval $1 \leq b \leq n\,a - 1$, with n a finite integer $n < \infty$. a can be small, medium, or unlimitedly large.

Adding a, I obtain $a+1 \leq c \leq (n+1)\,a-1$, and

$$k = \frac{c}{r^{1+\epsilon}} \leq \frac{(n+1)\,a-1}{a^{1+\epsilon}\,(\mathrm{rad}(b\,c))^{1+\epsilon}} \leq \frac{(n+1)\,a-1}{(2a)^{1+\epsilon}} < \frac{(n+1)}{2^{1+\epsilon}} < \infty\,, \tag{3}$$

as $\mathrm{rad}(b\,c) \geq 2$, and $\mathrm{rad}(a\,b\,c) = \mathrm{rad}(a)\,\mathrm{rad}(b\,c)$ for pairwise coprime numbers. Therefore, if b in the interval $1 \leq b \leq n\,a-1$ is any number, the abc conjecture holds. If a is unlimitedly large, the range for b goes from 1 to an unlimitedly large value.

Because the abc conjecture is formulated with $\epsilon \neq 0$, infinite many triplets (a,b,c) are at $k=0$. So, infinite many triplets are with $k<1$.

There are two areas: $k<1$ an $k>1$. The infinity is in $k<1$. So, there is no infinity in $k>1$.

5. No transitions between $k=0$ and $k=\infty$

In the previous part of the paper I have shown that there are infinitely many triplets for $k<1$. Therefore, if the abc conjecture fails and c is gradually increased, k starts to bounce endlessly between a value near zero and large values ($k \gg 1$). In terms of k, there are an infinite number of forth and back trans-passings, each of those leaves behind a trace of the triplets. Hence, an infinite number of triplets would be expected within the open interval $1<k<T$. An alternative formulation of the abc conjecture is that for $k>1$, there is a finite number of triplets [2]. Hence, the number of triplets within $1<k<T$ has to be finite. Otherwise, even if $K(\epsilon)<\infty$, the conjecture fails, because there is an infinite amount of triplets with $k>1$. But if $K(\epsilon)<\infty$, the conjecture cannot fail. We came to a contradiction. Hence, the number of triplets within $1<k<T$ is finite.

6. Final proof

We have

$$\kappa(c) = \frac{c}{(R(c))^{1+\epsilon}}\,, \quad \kappa(c-m) = \frac{c-m}{(R(c-m))^{1+\epsilon}}\,. \tag{4}$$

The ratio of latter two things is

$$\beta = \frac{\kappa(c-m)}{\kappa(c)} = \frac{c-m}{c}\left(\frac{R(c)}{R(c-m)}\right)^{1+\epsilon}\,, \tag{5}$$

where m is any integer from $1 \leq m \leq c/2$.

The existence of the limit $\lim_{c\to\infty} R(c) = \infty$, proven in Ref. [3], points to the inequality

$$\frac{R(c)}{R(c-m)} \geq L\,, \tag{6}$$

for every finite m, e.g., $m = 2, 3, 4, \ldots, 100$, constant $L \neq 0$ does not depend on c.

Let us assume for a moment that the abc conjecture fails. Because there are infinitely many triplets near $\kappa = 0$, such c, m exist, that $\kappa(c-m) \ll \kappa(c)$. So, $\beta = \frac{\kappa(c-m)}{\kappa(c)} \approx 0$, Therefore, $R(c)/R(c-m) \approx 0$ in the formula (5). But this is in contradiction with Eq. (6).

6.1. **The boundary of limit.** Let us define

$$Z = \frac{R(c+Y)}{R(c)} \frac{R(c)}{R(c-m)} = \frac{R(\hat{c}+m+Y)}{R(\hat{c})}\,, \tag{7}$$

where $\hat{c} = c - m$. Such an integer Y exists within $2 - c \leq Y < \infty$ so that

$$Z > G \tag{8}$$

together with

$$\frac{R(c+Y)}{R(c)} < M \tag{9}$$

because non-vanishing G can be arbitrarily small, and the finite M can be arbitrarily large. The $Y = Y(c)$.

Eqs. (7), (8), (9) imply

$$\frac{R(c)}{R(c-m)} > \frac{G}{M} = L\,. \tag{10}$$

References

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[3] C. L. Stewart, R. Tijdeman, "On the Oesterlé-Masser conjecture". Monatshefte für Mathematik. 102 (3): 251–257 (1986). doi:10.1007/BF01294603.