A curious family of integrals that give Pi

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ABSTRACT: In this note we give a set of integrals for Pi

KEYWORDS: Integrals for Pi, Sequence for Pi.

1. Introduction

The number Pi is defined by (Leibniz):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \tag{1}$$

The harmonic number is defined by:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\ ,\ n = 1, 2, 3, \dots ;\quad H_0 = 0 \tag{2}$$

The binomial coefficient is defined by:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\ ,\ n \ge k \ge 0 \tag{3}$$

The floor function is defined by:

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \le x\} \tag{4}$$

2. Family of integrals that give Pi

Define

$$a(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^n \binom{n}{2k}\ ,\ n = 1, 2, 3, \dots \tag{5}$$

$$b(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^n \binom{n}{2k+1}\ ,\ n = 1, 2, 3, \dots \tag{6}$$

For n = 1, 2, 3, ... we have

$$\pi = \frac{a(n)}{2^{n-2}} \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (-1)^{k-1} \binom{n}{2k-1} (H_n - H_{n-2k+1}) + \frac{b(n)}{2^{n-2}} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{k-1} \binom{n}{2k} (H_n - H_{n-2k}) + \frac{(-1)^n n!}{2^{n-2}} \int_0^\infty x^{-n-1} e^{-x} \left(a(n) \left(\sin x - \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right) + b(n) \left(\cos x - \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k x^{2k}}{(2k)!} \right) \right) dx \tag{7}$$

3. Examples

$$\pi = 2 + 2\int_0^\infty x^{-2}\, e^{-x}(1 + x - \cos x - \sin x)\, dx \tag{8}$$

$$\pi = 3 + 4\int_0^\infty x^{-3}\, e^{-x}\left(\operatorname{cox} x - 1 + \frac{x^2}{2}\right) dx \tag{9}$$

$$\pi = \frac{10}{3} - 6\int_0^\infty x^{-4}\, e^{-x}\left(\left(\cos x - 1 + \frac{x^2}{2}\right) - \left(\sin x - x + \frac{x^3}{6}\right)\right) dx \tag{10}$$

$$\pi = \frac{10}{3} - 24\int_0^\infty x^{-5}\, e^{-x}\left(\sin x - x + \frac{x^3}{6}\right) dx \tag{11}$$

$$\pi = \frac{97}{30} + 60\int_0^\infty x^{-6}\, e^{-x}\left(\left(\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}\right)\right) dx \tag{12}$$

$$\pi = \frac{63}{20} - 360\int_0^\infty x^{-7}\, e^{-x}\left(\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}\right) dx \tag{13}$$

$$\pi = \frac{109}{35} + 1260\int_0^\infty x^{-8}\, e^{-x}\left(\left(\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}\right) - \left(\sin x - x + \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040}\right)\right) dx \tag{14}$$

4. Elementary sequence

Define

$$u(n) = \frac{a(n)}{2^{n-2}} \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (-1)^{k-1} \binom{n}{2k-1} (H_n - H_{n-2k+1}) + \frac{b(n)}{2^{n-2}} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{k-1} \binom{n}{2k} (H_n - H_{n-2k}) \quad , \; n = 1, 2, 3, \ldots \tag{15}$$

we have

$$\lim_{n\to\infty} u(n) = \pi \tag{16}$$

5. References

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