Supportive intersection

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Abstract

Let X be a differential manifold. Let $\mathscr{D}'(X)$ be the space of currents, and $S^\infty(X)$ the Abelian group freely generated by regular cells, each of which is a pair of a polyhedron Π and a differential embedding of a neighborhood of Π to X. In this paper, we define a variant that is a bilinear map

$$\begin{array}{ccc} S^\infty(X)\times S^\infty(X) & \rightarrow & \mathscr{D}'(X) \\ (c_1, c_2) & \rightarrow & [c_1\wedge c_2] \end{array} \tag{0.1}$$

called the supportive intersection such that

1) the support of $[c_1\wedge c_2]$ is contained in the intersection of the supports of c_1, c_2;
2) if c_1, c_2 are closed, $[c_1\wedge c_2]$ is also closed and its cohomology class is the cup-product of the cohomology classes of c_1, c_2.

Contents

1 Introduction

This paper explores the notion of the support in differential topology. In general, the support is not an structural invariant, rather a property possessed by a

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structural invariant. So, throughout this paper, "support" or "supp" denotes the closed subset in the usual sense followed by various invariants such as singular chains, differential forms, currents etc.

Explicitly on a differentiable manifold X, we construct a variant that is a bilinear map

$$\begin{array}{ccc} S^\infty(X)\times S^\infty(X) & \to & \mathscr{D}'(X) \\ (c_1, c_2) & \to & [c_1\wedge c_2] \end{array} \tag{1.1}$$

such that

Condition 1.1. *(supportivity) the support of $[c_1\wedge c_2]$ is contained in the intersection of the supports of c_1, c_2;*

Condition 1.2. *(cohomologicality) if c_1, c_2 are closed, $[c_1\wedge c_2]$ is also closed and its cohomology class is the cup-product of the cohomology classes of c_1, c_2.*

The idea of the construction goes back to de Rham's work on currents. Originally in order to understand the homology of the complex of currents, de Rham constructed, for an arbitrary current T, the regularization $R_\epsilon T$ for a real number $\epsilon > 0$ such that the regularization weakly converges to T as $\epsilon \to 0$. The supportive property is a unique part of this regularization. In particular, it satisfies that

1) there exists another linear operator A_ϵ satisfying the homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT \tag{1.2}$$

where b is the boundary operator on currents,

2) the support of $R_\epsilon T$ is contained in any given neighborhood of the support of T provided ϵ is sufficiently small;

The development in the property 1) is well-known. But the implication of the property 2) has not been explored further.

In this paper, we are going to focus on the property 2). We work with singular chains which are known to be a particular type of currents. Let c be a chain, ω a smooth form. Denote the current of the integration over c by T_c, the current of the integration with ω by T_ω. If for any two regular chains c_1, c_2, i.e. those chains in $S^\infty(X)$, we can prove the existence of the weak limit of the current

$$T_{c_1}\wedge R_\epsilon c_2, \ as\ \epsilon\to 0 \tag{1.3}$$

then Conditions 1.1 and 1.2 easily follow from 1) and 2). The precise statement is our main theorem in the following.

Theorem 1.3. *(Main theorem) Let X be a differential manifold. For $(c_1, c_2) \in S^\infty(X) \times S^\infty(X)$, the weak limit*

$$\lim_{\epsilon \to 0} T_{c_1} \wedge R_\epsilon c_2, \tag{1.4}$$

exists in $\mathscr{D}'(X)$. Furthermore, the weak limit (1.4) denoted by $[c_1 \wedge c_2]$, called the supportive intersection satisfies Condition 1.1 and Condition 1.2.

Remark The $\lim\limits_{\epsilon \to 0} T_{c_1} \wedge R_\epsilon c_2$ is a weak limit in functional analysis and also extrinsically dependent of the regularization. But de Rham's original work in [2] is neither in functional analysis nor extrinsically dependent.

We organize the rest as follows. In Section 2, we prove a local property of de Rham's regularization R_ϵ. In Section 3, we prove that the existence of (1.4) follows from this property. In section 4, we verify that the limit (1.4) satisfies Conditions 1.1 and 1.2.

2 A property of de Rham's Regularization

Throughout the paper we denote the origin of the various Euclidean space $\mathbb{R}^\bullet$ by the same notation $\mathbf{0}$.

Definition 2.1. *(blow-up forms)*

Let $\digamma_\epsilon$ for $\epsilon > 0$ be a family of smooth forms of degree r in an Euclidean space $\mathbb{R}^m$. If there are an orthogonal decomposition $\mathbb{R}^m = \mathbb{R}^r \times \mathbb{R}^{m-r}$ with coordinate $\mathbf{u}$ for the subspace $\mathbb{R}^r \times \{\mathbf{0}\}$ and a smooth form $\digamma_1(\mathbf{u})$ on $\mathbb{R}^r \times \{\mathbf{0}\}$ with a compact support such that

$$\digamma_\epsilon = \pi^* \digamma_1(\frac{\mathbf{u}}{\epsilon}) \tag{2.1}$$

or abbreviated as

$$\digamma_\epsilon = \digamma_1(\frac{\mathbf{u}}{\epsilon})$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^r \times \{\mathbf{0}\}$ is the orthogonal projection, then $\digamma_\epsilon$ is called a blow-up form from $\digamma_1(\mathbf{u})$ along $\mathbb{R}^r \times \{\mathbf{0}\}$ at $\{\mathbf{0}\} \times \mathbb{R}^{m-r}$.

One of the main techniques in [2] is the construction of de Rham's regularization for currents. In today's standard, the details may be outdated. However, it is quite unique among other types of regularization.

Theorem 2.2. *(G. de Rham) Let ϵ be a small positive number. Let $\mathscr{E}(X)$ be the space of smooth forms on X. Then there exist linear operators on X,*

$$\begin{array}{rcl} R_\epsilon : \mathscr{D}'(X) & \to & \mathscr{E}(X) \\ A_\epsilon : \mathscr{D}'(X) & \to & \mathscr{D}'(X) \end{array} \tag{2.2}$$

such that for $T \in \mathscr{D}'(X)$

(1) a homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT, \tag{2.3}$$

holds where b is the boundary operator,

(2) $supp(R_\epsilon T), supp(A_\epsilon T)$ are contained in any given neighborhood of $supp(T)$ provided ϵ is sufficiently small,

(3) If a smooth differential form ϕ has the bounded semi-norm $||\bullet||_{q,K}$ where q is a whole number and K is a compact set and ϵ is bounded above, then $R_\epsilon T_\phi, A_\epsilon T_\phi$ are also bounded in the same semi-norm,

(4)

$$\lim_{\epsilon\to 0} R_\epsilon T = T, \quad \lim_{\epsilon\to 0} A_\epsilon T = 0$$

in the weak topology of $\mathscr{D}'(X)$. Furthermore, the convergence is uniform on the set of forms with the bounded semi-norms $|| \bullet ||_{q,K}$.

The collection of the data used in the regularization is called de Rham data. In particular, it consists of countably many ordered open sets $U_1, \cdots$ where the local regularization occur independently in each U_i. The global regularization R_ϵ is just the iteration of the local regularization.

This paper only needs the properties (1) and (2). In addition, we recall another notion by de Rham: smooth kernel. Let

$$\Lambda : \mathscr{D}'(X) \to \mathscr{D}'(X)$$

be a linear operator on currents. The operator is said to be *regularizing* if $\Lambda(\mathscr{D}'(X))$ is contained in the subset $\mathscr{E}(X) \subset \mathscr{D}'(X)$. This assumption implies that there is a smooth form L on $X \times X$ such that for $\phi \in \mathscr{D}(X)$ and $T \in \mathscr{D}'(X)$ with a compact support

$$\Lambda(T)[\phi] = (T \otimes T_\phi)[L].$$

The form L is called the smooth kernel of the operator Λ. It is known that de Rham's regulator is regularizing, therefore it has a smooth kernel. In the following, we prove a local property of this smooth kernel.

Proposition 2.3.

At each point of X, there is a neighborhood $U \simeq \mathbb{R}^m$, such that the smooth kernel of de Rham's regulator R_ϵ with sufficiently small ϵ is restricted to a blow-up form on $\mathbb{R}^m \times \mathbb{R}^m$ at the diagonal Δ, where $\mathbb{R}^m$ is the Euclidean space diffeomorphic to U.

Proof. We need to analyze the local structure of the regulator. So, we start with the reviewing of the de Rham's construction in its local charts. Let $\mathbb{R}^m$ be the Euclidean space of dimension m with a linear structure. Let $y_1, \cdots, y_m$ be its coordinates under a basis. They will be collectively denoted by the **bold** letter $\mathbf{y}$. Same **bold** fonts for various Euclidean spaces will be used throughout this paper. Let $f(\mathbf{y}) \in \mathscr{D}(\mathbb{R}^m)$ be a function (i.e. a mollifier) supported in the unit ball such that

$$\int_{\mathbf{y}\in\mathbb{R}^m} f(\mathbf{y}) d\mu_{\mathbf{y}} = 1,$$

where $d\mu_{\mathbf{y}}$ is the volume form

$$dy_1 \wedge \cdots \wedge dy_m.$$

Let

$$\vartheta_\epsilon(\mathbf{y}) = \frac{1}{\epsilon^m} f(\frac{\mathbf{y}}{\epsilon}) d\mu_{\mathbf{y}}, \epsilon > 0 \tag{2.4}$$

be the m-form on $\mathbb{R}^m$. Then the de Rham's regulator on $\mathbb{R}^m$ is the operator that sends each current T on $\mathbb{R}^m$ to the form

$$\pm T[\vartheta_\epsilon\big(\mathbf{x}-\mathbf{y}\big)]_{\mathbf{y}} \tag{2.5}$$

where the sign $\pm$ is determined by the dimension of T and m, the current T is evaluated at the form of $\mathbf{y}$ variable. The operator depends on the coordinates of $\mathbb{R}^m$. We denote this regulator by $\mathcal{R}_\epsilon$. The form

$$\pm\vartheta_\epsilon\big(\mathbf{x}-\mathbf{y}\big) \tag{2.6}$$

on $\mathbb{R}^m \times \mathbb{R}^m$ is denoted by $\theta_\epsilon(\mathbf{x}-\mathbf{y})$ where $\mathbf{x}, \mathbf{y}$ are the variables for the first and second factors in $\mathbb{R}^m \times \mathbb{R}^m$. Notice that $\theta_\epsilon(\mathbf{x}-\mathbf{y})$ is the smooth kernel of $\mathcal{R}_\epsilon$ (with respect to the degree of c). The extension to the global X is through a countable iteration of the local $\mathcal{R}_\epsilon$. The extension requires countably many local charts $U_i \simeq \mathbb{R}^m$ in de Rham data that covers X. The covering is locally finite. By the continuity, we may only consider the point q not on the boundaries of U_i. Such an extension at the point q can be described as follows. Since the de Rham's covering is locally finite, there are finitely many ordered open sets, $U_1, U_2, \cdots, U_n$ that contain q. It suffices to consider the regularization in these open sets. We denote the regulator on each U_i by $\mathcal{R}^i_\epsilon$ and its smooth kernel by $\theta^i_\epsilon(\mathbf{x}_i - \mathbf{y}_i)$. By the partition of unity, we may only consider the current T compactly supported in the overlap $\bigcap\limits_i U_i$. Then the global R_ϵ sends the T to a smooth form

$$\mathcal{R}^n_\epsilon \circ \mathcal{R}^{n-1}_\epsilon \circ \cdots \circ \mathcal{R}^1_\epsilon(T). \tag{2.7}$$

Above is the description of de Rham's construction around the point q. The following is our work to show that the kernel of (2.7) is a blow-up form. First we'll express the kernel. In each local regulator

$$\mathcal{R}^i : \mathscr{D}'(\mathbb{R}^m) \to \mathscr{D}'(\mathbb{R}^m)$$

we denote the $\mathbb{R}^m$ in the domain space by $\mathbb{R}^m_{\mathbf{y}_i}$ with the variable $\mathbf{y}_i$, and the $\mathbb{R}^m$ in the target space by $\mathbb{R}^m_{\mathbf{x}_i}$ with variable $\mathbf{x}_i$. We identify $\mathbb{R}^m_{\mathbf{y}_i} = \mathbb{R}^m_{\mathbf{x}_{i-1}}$ and denote the same space by $\mathbb{R}^m_{i,(i-1)}$ (which is a copy of $\mathbb{R}^m$). Then each product $\mathbb{R}^m_{\mathbf{x}_i} \times \mathbb{R}^m_{\mathbf{y}_i}$ for $i = n, \cdots, 1$ is embedded in

$$\mathbb{R}^m_{\mathbf{x}_n} \times \mathbb{R}^m_{n,n-1} \times \mathbb{R}^m_{n-1,n-2} \times \cdots \times \mathbb{R}^m_{2,1} \times \mathbb{R}^m_{\mathbf{y}_1}$$

as the zero-section of the trivial bundle. So, we pull back each $\theta^i_\epsilon(\mathbf{x}_i - \mathbf{y}_i)$ to the the product

$$\mathbb{R}^m_{\mathbf{x}_n} \times \mathbb{R}^m_{n,n-1} \times \mathbb{R}^m_{n-1,n-2} \times \cdots \times \mathbb{R}^m_{2,1} \times \mathbb{R}^m_{\mathbf{y}_1}$$

and denote the pullback by the same notation $\theta^i_\epsilon(\mathbf{x}_i - \mathbf{y}_i)$. Then according to (2.7), the local expression of the global kernel $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is the fibre integral

$$\int_{(\mathbf{y}_n,\cdots,\mathbf{y}_2)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta^n_\epsilon(\mathbf{x}_n - \mathbf{y}_n) \wedge \theta^{n-1}_\epsilon(\mathbf{x}_{n-1} - \mathbf{y}_{n-1}) \wedge \cdots \wedge \theta^2_\epsilon(\mathbf{x}_2 - \mathbf{y}_2) \wedge \theta^1_\epsilon(\mathbf{x}_1 - \mathbf{y}_1), \quad (2.8)$$

where $\theta^i_\epsilon(\mathbf{x}_i - \mathbf{y}_i)$ is the smooth kernel of $\mathcal{R}_i$. So the global kernel $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is a m-form on the product

$$\mathbb{R}^m_{\mathbf{x}_n} \times \mathbb{R}^m_{\mathbf{y}_1} = \mathbb{R}^m \times \mathbb{R}^m$$

where $\mathbf{x}_n, \mathbf{y}_1$ are the coordinates for the first and second factor of the kernel. In (2.8), we define the new coordinates:

$$\mathbf{w}_i = \mathbf{x}_i - \mathbf{y}_i \quad (2.9)$$

where $i = 1, \cdots, n-1$, also

$$\mathbf{x}_n - \mathbf{y}_1 - \Big(\mathbf{w}_1 + \cdots + \mathbf{w}_{n-1}\Big) = \mathbf{x}_n - \mathbf{y}_n. \quad (2.10)$$

Then (2.8) is equal to

$$\int_{(\mathbf{w}_{n-1},\cdots,\mathbf{w}_1)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta^n_\epsilon\Big(\mathbf{x}_n - \mathbf{y}_1 - (\mathbf{w}_1 + \cdots + \mathbf{w}_{n-1})\Big) \wedge \theta^{n-1}_\epsilon(\mathbf{w}_{n-1}) \wedge \cdots \wedge \theta^1_\epsilon(\mathbf{w}_1), \quad (2.11)$$

Divide each variable by ϵ, we obtain that $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is equal to

$$\int_{(\mathbf{w}_{n-1},\cdots,\mathbf{w}_1)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta^n_1\Big(\frac{\mathbf{x}_n - \mathbf{y}_1}{\epsilon} - (\mathbf{w}_1 + \cdots + \mathbf{w}_{n-1})\Big) \wedge \theta^{n-1}_1(\mathbf{w}_{n-1}) \wedge \cdots \wedge \theta^1_1(\mathbf{w}_1). \quad (2.12)$$

So, if we denote the m form on $\mathbb{R}^m$,

$$\int_{(\mathbf{w}_{n-1},\cdots,\mathbf{w}_1)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta_1^n\left(\frac{\mathbf{z}}{\epsilon} - \left(\mathbf{w}_1 + \cdots + \mathbf{w}_{n-1}\right)\right) \wedge \theta_1^{n-1}(\mathbf{w}_{n-1}) \wedge \cdots \wedge \theta_1^1(\mathbf{w}_1) \quad (2.13)$$

by $F_\epsilon(\mathbf{z})$ for the variable $\mathbf{z}$ of $\mathbb{R}^m$, then

$$\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1) = \kappa^* F_\epsilon \quad (2.14)$$

where κ is the map: $(\mathbf{x}_n, \mathbf{y}_1) \to \mathbf{x}_n - \mathbf{y}_1$. Since all forms $\theta_1^j(\mathbf{z}), j = n, \cdots, 1$ have compact supports, so $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is a blow-up form from a compactly supported form F_1. We complete the proof.

□

3 Convergence in de Rham's regularization

The main technical result is the following proposition about the convergence. It concerns a particular type of de Rham's wedge products between a cell and a form.

Proposition 3.1. *Let c be a p dimensional regular cell in $\mathbb{R}^m$. Let ω_ϵ be a blow-up form of degree $r \leq p$ in $\mathbb{R}^m$. Then the current*

$$T_c \wedge \omega_\epsilon \quad (3.1)$$

converges weakly to a current as $\epsilon \to 0$.

The special case of the proposition for the blow-up at a point is used in cohomology theory (see Example 3.4 below). But the general case for the blow-up at a higher dimensional subspace has not been looked at. In the technique, our central idea is to convert the convergence of integrals of forms to that of sets. Wildly behaved sets usually do not respect arithmetic and not even manifold's structures, but they are effectively used in the foundation of probability and measure theory.

Notice that the convergence only concerns the local Euclidean space and one cell in it. So we focus on the Euclidean space. Throughout, for an Euclidean space $\mathbb{R}^l$ with a coordinate $\mathbf{z}$, we'll abuse the notation to denote the volume form of a subspace with the concordant orientation and the volume density in Lebesgue integrals by the same expression $d\mu_{\mathbf{z}}$. The argument starts with a definition and a lemma about points and sets.

Definition 3.2. *Let $W \subset \mathbb{R}^p$ be a subset in an Euclidean space with the origin $\mathbf{o}$. A point $\mathbf{a} \in \mathbb{R}^p$ is said to be a stable point of W if the line segment*

$$\{\mathbf{o} + t(\overrightarrow{\mathbf{oa}}),\ 0 < t \leq 1\}$$

either lies in W completely or in W^c completely, where $\overrightarrow{\mathbf{oa}} \in T_{\mathbf{o}}\mathbb{R}^p = \mathbb{R}^p$ is the vector from $\mathbf{o}$ to $\mathbf{a}$, and W^c is the complement $\mathbb{R}^p \backslash W$. We denote the collection of stable points of W by $W_s^{\mathbf{o}}$.

Let $c : \Pi_p \to \mathbb{R}^m$ be a regular cell as in Proposition 3.1 with the p-dimensional polyhedron Π_p. Let $C = c(\Pi_p)$ be the image of the cell. Let $\mathbb{R}^r, \mathbb{R}^{p-r}, \mathbb{R}^{m-p}$ be subspaces with coordinates $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively such that

$$\mathbb{R}^m = \mathbb{R}^r \times \mathbb{R}^{p-r} \times \mathbb{R}^{m-p}. \tag{3.2}$$

Let

$$\eta : \mathbb{R}^m \quad \to \quad \mathbb{R}^p = \mathbb{R}^r \times \mathbb{R}^{p-r} \times \{\mathbf{0}\}$$

be the projection to its subspace $\mathbb{R}^p$. Let $D_{\frac{1}{\epsilon}}$ for a positive ϵ be the linear transformation of $\mathbb{R}^m$ defined by the map

$$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) \quad \to \quad (\frac{\mathbf{u}}{\epsilon}, \mathbf{v}_1, \mathbf{v}_2). \tag{3.3}$$

In the context, we denote its restriction to subspaces also by $D_{\frac{1}{\epsilon}}$. All measures in the following are the Lebesgue measures on Euclidean spaces.

Lemma 3.3. *Denote $W := \eta(C)$. There exists a subset $W_{fu} \subset W$ of measure 0 such that the set-theoretic limit (§4, [1])*

$$\lim_{\epsilon \to 0} D_{\frac{1}{\epsilon}} \left(W \backslash W_{fu} \right) \tag{3.4}$$

exists *.

Proof. We denote

$$L := \{\mathbf{0}\} \times \mathbb{R}^{p-r} \times \{\mathbf{0}\}$$

For $\mathbf{o} \in L$, $\mathbf{o} = (\mathbf{0}, \mathbf{v}_1, \mathbf{0})$ is the origin for the partial scalar multiplication $D_{\frac{1}{\epsilon}}$. Let

$$W^{\mathbf{o}} = W \cap \left(\mathbb{R}^r \times \{\mathbf{v}_1\} \times \{\mathbf{0}\} \right).$$

*For a family of sets S_ϵ, the existence of the set-theoretic limit means

$$\bigcap_{\epsilon_1 \leq 1} \bigcup_{\epsilon_2 \leq \epsilon_1} S_{\epsilon_2} = \bigcup_{\epsilon_1 \leq 1} \bigcap_{\epsilon_2 \leq \epsilon_1} S_{\epsilon_2}$$

Therefore $\mathbf{o}$ is the origin of the affine plane $\mathbb{R}^r \times \{\mathbf{v}_1\} \times \{\mathbf{0}\}$. Let $\mathcal{R}_{\mathbf{o}}$ be the ray

$$\{\mathbf{o} + t(\overrightarrow{\mathbf{oa}}) : \mathbf{a} \in W^{\mathbf{o}}, t > 0\}$$

that starts at the origin in the affine plane. Let

$$W^{\mathbf{o}}_{fu} \subset W^{\mathbf{o}}$$

denote the subset

$$\{\mathbf{a} \in W^{\mathbf{o}} : \mathcal{R}_{\mathbf{o}} \textit{ does not contain a stable point of } W^{\mathbf{o}}\}.$$

We divide W to three disjoint parts.

1) $W_{fu} = \underset{\mathbf{o}\in L}{\cup} W^{\mathbf{o}}_{fu}$, called the set of fully unstable points,

2) $W_s = \underset{\mathbf{o}\in L}{\cup} W^{\mathbf{o}}_s$, called the set of stable points,

3) W_{pu} is $W\backslash(W_{fu} \cup W_s)$, called the set of partially unstable points.

Next we should dilate each part by a scalar multiplication $D_{\frac{1}{\epsilon}}$.

For the fully unstable points W_{fu}, we would like to show they are necessarily on the "boundary" which gives the measure 0. The following is the detail. The boundary of the polyhedron Π_p is defined by multiple hyperplanes. Hence the boundary of C is also defined by multiple hyperplanes H_j. On the other hand in the its target space, we let

$$\begin{array}{rcc} \nu : \mathbb{R}^r\backslash\{\mathbf{0}\} \times \mathbb{R}^{p-r} \times \{\mathbf{0}\} & \to & \mathbb{P}^{r-1} \times \mathbb{R}^{p-r} \times \{\mathbf{0}\} \\ (\mathbf{u}, \mathbf{v}_1, \mathbf{0}) & \to & ([\mathbf{u}], \mathbf{v}_1, \mathbf{0}) \end{array} \tag{3.5}$$

be the map that is the product of the projectivization map and the identity map (where $\mathbb{P}^{r-1}$ can be regarded as the real projectivization of $T_{\mathbf{0}}\mathbb{R}^r$, the set of directions). Fix a point $\mathbf{o} \in L$. Let $\mathbf{a} \in W^{\mathbf{o}}_{fu}$ other than $\mathbf{o}$. Since $\mathbf{a}$ is a fully unstable point, there are two sequences of points $\mathbf{p}_n, \mathbf{q}_n$ on the ray $\mathcal{R}_{\mathbf{o}}$ such that

$$\lim_{n\to\infty} \mathbf{p}_n = \mathbf{o} = \lim_{n\to\infty} \mathbf{q}_n$$

and

$$\mathbf{p}_n \notin W^{\mathbf{o}}, \mathbf{q}_n \in W^{\mathbf{o}}.$$

Thus the directions $\overrightarrow{\mathbf{op}_n}$ and $\overrightarrow{\mathbf{op}_n}$, which are all parallel to the tangent vector $\overrightarrow{\mathbf{oa}}$ must lie on at least one nontrivial plane $\eta_*(H_j)$. Since a subplane properly contained in an Euclidean space has a measure 0, for each fixed $\mathbf{o}$, $\mathbb{P}(W^{\mathbf{o}}_{fu}\backslash\{\mathbf{0}\})$ has measure 0 in the manifold

$$\mathbb{P}(\mathbb{R}^r\backslash\{\mathbf{0}\}) \times \{\mathbf{v}_1\} \times \{\mathbf{0}\} \simeq \mathbb{P}^{r-1}$$

where $\mathbf{v}_1$ is fixed. Since

$$\mathbb{R}^r\backslash\{\mathbf{0}\} \to \mathbb{P}^{r-1}$$

is a bundle's projection, the inverse $W_{fu}^{\mathbf{o}}$ also has measure 0. To go further, we take the union over L to obtain $\nu(W_{fu}\backslash L) = \underset{\mathbf{o}\in L}{\cup}\mathbb{P}(W_{fu}^{\mathbf{o}}\backslash\{\mathbf{o}\})$ has measure 0 in the manifold

$$\mathbb{P}^{r-1}\times\mathbb{R}^{p-r}\times\{\mathbf{0}\}.$$

Due to the fibre bundle structure of the projectivization, we conclude W_{fu} in $\mathbb{R}^p$ has measure 0. Notice that $D_{\frac{1}{\epsilon}}$ is a linear transformation, $D_{\frac{1}{\epsilon}}(W_{fu})$ which is equal to W_{fu} also has measure 0. Therefore the limit is of 0. †

For W_s, we consider the set $B_\epsilon = D_{\frac{1}{\epsilon}}(W_s)$. We would like to show B_ϵ as $\epsilon\to 0$ is a decreasing set. Let $\mathcal{R}_{\mathbf{o}}$ be the ray starting at $\mathbf{o}\in L$ and through a stable point $\mathbf{a}\in W_s^{\mathbf{o}}$ of $W^{\mathbf{o}}$ for an $\mathbf{o}\in L$. Since $\mathbf{a}$ is stable, the dilation by the scalar multiplication $D_{\frac{1}{\epsilon}}$ yields

$$D_{\frac{1}{\epsilon}}(\mathcal{R}_{\mathbf{o}}\cap W_s)\subset D_{\frac{1}{\epsilon'}}(\mathcal{R}_{\mathbf{o}}\cap W_s),\quad \text{for } \epsilon'<\epsilon<1.$$

Now taking the union over all the rays through stables points, we obtain

$$D_{\frac{1}{\epsilon}}(W_s)\subset D_{\frac{1}{\epsilon'}}(W_s),\quad \text{for } \epsilon'<\epsilon.$$

Therefore B_ϵ is a decreasing family of measurable sets. Let

$$B_0 := \cup_{\epsilon\in(0,1]}\bigg(D_{\frac{1}{\epsilon}}(W_s)\bigg). \tag{3.6}$$

Then set-theoretically the decreasing family yields

$$\lim_{\epsilon\to 0} B_\epsilon = B_0$$

and B_0 is measurable.

For W_{pu}, we consider the set $A_\epsilon = D_{\frac{1}{\epsilon}}(W_{pu})$. We would like to show A_ϵ as the set multiplied by $\dfrac{1}{\epsilon}$ will be pushed to ∞ as $\epsilon\to 0$. Here is the detail. If $\bigcap\limits_{\epsilon_1\le 1}\bigcup\limits_{\epsilon_2\le\epsilon_1} A_{\epsilon_2}$ is non-empty, there is a point

$$\mathbf{x}\in\bigcap_{\epsilon_1\le 1}\bigcup_{\epsilon_2\le\epsilon_1} A_{\epsilon_2}$$

i.e. $\mathbf{x}\in\bigcup\limits_{\epsilon_2\le\epsilon_1} A_{\epsilon_2}$ for any $\epsilon_1<1$. So, there is a sequence of numbers ϵ_n such that $\lim\limits_{n\to\infty}\epsilon_n = 0$ and $D_{\epsilon_n}(\mathbf{x})$ lies in W_{pu}. Suppose that N is a number in the sequence such that $D_{\epsilon_N}(\mathbf{x})\in W_{pu}$. By the definition of W_{pu}, there is a smaller $\epsilon_{N'}\neq 0$ such that $D_{\epsilon_{N'}}(\mathbf{x})$ is a stable point, i.e. $D_{\epsilon_{N'}}(\mathbf{x})\in W_S$. Then all points $D_{\epsilon_n}(\mathbf{x})$ are stable whenever $\epsilon_n<\epsilon_{N'}$. But this contradicts the assertion above: there is a sequence of partially unstable points $\epsilon_n\mathbf{x}$ with $\epsilon_n\to 0$. Thus

†But the set W_{fu} is not on the boundary of W.

$$\lim_{\epsilon\to 0} sup A_\epsilon = \bigcap_{\epsilon_1\leq 1} \bigcup_{\epsilon_2\leq\epsilon_1} A_{\epsilon_2} = \varnothing. \tag{3.7}$$

Therefore

$$\lim_{\epsilon\to 0} inf A_\epsilon \subset \lim_{\epsilon\to 0} sup A_\epsilon$$

is also empty. Hence $\lim_{\epsilon\to 0} A_\epsilon$ exists and is equal to an empty set.

Combining the results for W_{fu}, W_s and W_{pu}, we complete the proof.

□

Proof of Proposition 3.1. We continue with all notations in Lemma 3.3. Let ϕ be a test form of degree $p-r$ in $\mathbb{R}^m$. It amounts to show the convergence of the integral

$$\int_C \omega_\epsilon \wedge \phi \tag{3.8}$$

as $\epsilon \to 0$, where $C = c(\Pi_p)$ is the image of the regular cell $c : \Pi_p \to \mathbb{R}^m$. Lemma 3.3 holds for any orthogonal decomposition (3.2) of $\mathbb{R}^m$. But, for Proposition 3.1, we have to specify the decomposition that needs to be related to the blow-up form. Let $\mathbb{R}^r \times \{\mathbf{0}\} \times \{\mathbf{0}\}$ be the subspace with coordinates $\mathbf{u}$ such that the blow-up form is written as

$$\omega_\epsilon = \frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon}) d\mu_{\mathbf{u}} \tag{3.9}$$

where $g(\mathbf{u})$ is a C^∞ function of $\mathbb{R}^r \times \{\mathbf{0}\} \times \{\mathbf{0}\}$. Notice that the form $\omega_\epsilon \wedge \phi$ is the sum of simple forms in the coordinates of $\mathbb{R}^m$ that can be explicitly expressed. So, we'll focus on the integral of a single simple form.

We work with the simple form written as

$$\frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon}) \psi(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{3.10}$$

where the volume forms $d\mu_{\mathbf{u}}, d\mu_{\mathbf{v}_1}$ determine two coordinate's planes

$$\mathbb{R}^r \times \{\mathbf{0}\} \times \{\mathbf{0}\}, \{\mathbf{0}\} \times \mathbb{R}^{p-r} \times \{\mathbf{0}\}$$

with coordinates $\mathbf{u}, \mathbf{v}_1$ respectively, and ψ is a C^∞ function on

$$\mathbb{R}^m = \mathbb{R}^r \times \mathbb{R}^{p-r} \times \mathbb{R}^{m-p}$$

that is the coefficient of the simple form $\psi d\mu_{\mathbf{v}_1}$ in a general test ϕ. Then the integral of (3.10) over C is

$$\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u}) \psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{3.11}$$

where $\mathbf{u}$ is the new variable obtained from the old $\mathbf{u}$ divided by ϵ. Let K_1 be the support of $g(\mathbf{u})$, and K_2, K_3 be the bounded sets of $\mathbb{R}^{p-r}, \mathbb{R}^{m-r}$ such that

C is contained in $\mathbb{R}^r \times K_2 \times K_3$. Then $\psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)$ uniformly converges to $\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$ in $K_1 \times K_2 \times K_3$. So, for any positive δ, we can find sufficiently small ϵ such that

$$|\psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) - \psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)| \leq \delta. \tag{3.12}$$

Let c_ϵ be the composition

$$\overline{\Pi}_p \xrightarrow{c} \mathbb{R}^m \xrightarrow{D_{\frac{1}{\epsilon}}} \mathbb{R}^m. \tag{3.13}$$

Notice

$$D_{\frac{1}{\epsilon}}(C) \cap (K_1 \times K_2 \times \times K_3)$$

is a bounded set. Thus all coefficients of the form $c_\epsilon^*\big(g(u)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1}\big)$ are bounded uniformly for all sufficiently small ϵ. Hence

$$\begin{aligned}&|\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} - \int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1}|\\ &\leq \delta M\end{aligned} \tag{3.14}$$

where M is a constant. For the integral

$$\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{3.15}$$

we make a change of variable from $\mathbf{u}$ to $\dfrac{\mathbf{u}}{\epsilon}$ to find (3.15) is equal to

$$\frac{1}{\epsilon^r}\int_C g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{3.16}$$

Now we apply Lemma A.1, there is a compactly supported integrable function $\tilde{\xi}_\epsilon(\mathbf{u}, \mathbf{v}_1)$ on $\mathbb{R}^p$ such that

$$\frac{1}{\epsilon^r}\int_C g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} = \frac{1}{\epsilon^r}\int_W g(\frac{\mathbf{u}}{\epsilon})\tilde{\xi}_\psi(\mathbf{u}, \mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \tag{3.17}$$

where W is the measurable set defined in Lemma 3.3, and the right hand side is a Lebesgue integral with the density measure $d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$, and $\tilde{\xi}_\psi(\mathbf{u}, \mathbf{v}_1)$ in the integrand is a compactly supported L^1 function on $\mathbb{R}^p$. Furthermore, since $\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$ is a pullback function from $\{\mathbf{0}\} \times \mathbb{R}^{p-r} \times \mathbb{R}^{m-r}$, then $\tilde{\xi}_\psi(\mathbf{u}, \mathbf{v}_1)$ is also a pullback of function $\xi_\psi(\mathbf{v}_1)$ from $\{\mathbf{0}\} \times \mathbb{R}^{p-r} \times \{\mathbf{0}\}$. So, in the following, we express the pullback function $\tilde{\xi}_\psi(\mathbf{u}, \mathbf{v}_1)$ as $\xi_\psi(\mathbf{v}_1)$. Now changing the variables from $\dfrac{\mathbf{u}}{\epsilon}$ back to $\mathbf{u}$, we have

$$\begin{aligned}(3.17) &= \textstyle\int_{\mathbb{R}^p} \chi_{D_{\frac{1}{\epsilon}}(W)}(\mathbf{u}, \mathbf{v}_1)g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}\\ &= \textstyle\int_{\mathbb{R}^p} \chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}(\mathbf{u}, \mathbf{v}_1)g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}\end{aligned} \tag{3.18}$$

where $\chi_\bullet$ denotes the characteristic function of the set $\bullet$. Next for the Lebesgue integrals, we'll omit the notations for variables for the dominant convergence theorem. We'll see that the integrand in (3.18) satisfies

$$|\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})} g\xi_\psi| \le |g\xi_\psi|$$

and $|g\xi_\psi|$ is an L^1 function on $\mathbb{R}^p$. The set-theoretic convergence in Lemma 3.3 implies the $\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})} g\xi_\psi$ point-wisely converges to the function

$$\chi_{B_0} g\xi_\psi.$$

By the dominant convergence theorem

$$\begin{aligned} &\lim_{\epsilon\to 0}\int_{\mathbb{R}^p} \chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})} g\xi_\psi d\mu_{\mathbf{u}} d\mu_{\mathbf{v}_1} = \int_{\mathbb{R}^p} \chi_{B_0} g\xi_\psi d\mu_{\mathbf{u}} d\mu_{\mathbf{v}_1} \\ &= \int_{B_0} g\xi_\psi d\mu_{\mathbf{u}} d\mu_{\mathbf{v}_1} \end{aligned} \tag{3.19}$$

Finally, combining (3.14) and (3.19), we obtain that

$$\lim_{\epsilon\to 0}\int_C \frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} = \int_{B_0} g(\mathbf{u})\xi_\psi(\mathbf{v}_1) d\mu_{\mathbf{u}} d\mu_{\mathbf{v}_1}$$

(Note the right hand side is an integral of a differential form but the left is a Lebegue integral). We conclude

$$T_c \wedge \omega_\epsilon$$

converges to a functional as $\epsilon \to 0$. For the continuity of the functional, we see that if ϕ is bounded to any orders, then in particular ϕ is bounded to the order of 0. Hence the formula (3.15) is bounded. Then

$$\int_{B_0} g(\mathbf{u})\xi_\psi(\mathbf{u}, \mathbf{v}_1) d\mu_{\mathbf{u}} d\mu_{\mathbf{v}_1}$$

is bounded. So, the evaluation

$$\lim_{\epsilon\to 0}(T_c \wedge \omega_\epsilon)[\phi]$$

is also bounded. Hence the functional

$$\phi \to \lim_{\epsilon\to 0}(T_c \wedge \omega_\epsilon)[\phi]$$

defines a current. The proof is completed.

□

Remark The particular type of wedge products in (3.1) provided the early proof for the homology of the complex of currents (see Chapter IV, [2]). As the new technique emerged, this type of analysis is no longer necessary. However, some still remain. The following example is the remaining case in cohomology theory (see section 1, Chapter 3 in [3]).

Example 3.4. *Let c be an n-dimensional polyhedron in $\mathbb{R}^n$ and contain the origin. Let ω_ϵ be a blow-up form at the origin with the top degree n (i.e. along the entire space $\mathbb{R}^n$). Then as $\epsilon \to 0$, $T_c \wedge \omega_\epsilon$ converges weakly to a constant multiple of δ function at the origin. The constant multiple is 1 if ω_ϵ is normalized.*

The following proposition proves the first part of Main theorem 1.3.

Proposition 3.5. *Let X be a differential manifold of dimension m. For chains c_1, c_2 in $S^\infty(X)$, the exterior product*

$$T_{c_1} \wedge R_\epsilon c_2 \tag{3.20}$$

converges weakly to a current as $\epsilon \to 0$.

Proof. It suffices to assume $c_2 : \Pi_p \to \mathbb{R}^m$ is a regular cell and it lies in an open neighborhood U as in Proposition 2.3. We subdivide c_1 to a sum of smaller regular cells so that there are finitely many regular cells σ_j that cover the $supp(R_\epsilon c_2)$ for sufficient small ϵ and $supp(\sigma_j) \subset U$. Then

$$T_{c_1} \wedge R_\epsilon c_2 = \sum_j T_{\sigma_j} \wedge R_\epsilon c_2.$$

So, it suffices to prove the proposition for c_1 whose support lies in U. For a test form ϕ, the evaluation

$$\left(T_{c_1} \wedge R_\epsilon c_2\right)[\phi]$$

is equal to the integral in $X \times X$ as

$$\int_{(\mathbf{x},\mathbf{y}) \in c_1 \times c_2} \varrho_\epsilon(\mathbf{x}, \mathbf{y}) \wedge P^*(\phi)(\mathbf{x}) \tag{3.21}$$

where $P : X \times X \to X(1st\ copy)$ is the projection, $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ is the kernel of R_ϵ. By Proposition 2.3, $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ is a blow-up form in the Euclidean space $U \times U \simeq \mathbb{R}^{2m}$ at the diagonal Δ_U. Thus (3.21) is the evaluation of the current

$$T_{c_1 \times c_2} \wedge \varrho_\epsilon(\mathbf{x}, \mathbf{y}) \tag{3.22}$$

at a particular form $P^*(\phi)$. By Proposition 3.1, the limit

$$\lim_{\epsilon \to 0} \int_{(\mathbf{x},\mathbf{y}) \in c_1 \times c_2} \varrho_\epsilon(\mathbf{x}, \mathbf{y}) \wedge P^*(\phi)(\mathbf{x}) \tag{3.23}$$

exists, and bounded by $||\phi||_\infty$. Hence

$$\lim_{\epsilon \to 0} T_{c_1} \wedge R_\epsilon c_2$$

is a current. The proof is completed.

□

4 The supportive intersection

Definition 4.1. *Let X be a differential manifold. Let c_1, c_2 be two chains in $S^\infty(X)$. We define*

$$[c_1 \wedge c_2]$$

to be the weak limit

$$\lim_{\epsilon \to 0} \big(T_{c_1} \wedge R_\epsilon c_2 \big).$$

It gives a rise to a well-defined bilinear map

$$S^\infty(X) \times S^\infty(X) \to \mathscr{D}'(X).$$

We call the map the supportive intersection.

The following properties (1) and (2) complete the proof for the second part of Main theorem 1.3.

Property 4.2.
Let X a differential manifold of dimension m. For chains c_1, c_2 in $S^\infty(X)$, the supportive intersection $[c_1 \wedge c_2]$ satisfies:

(1) (Supportivity)

$$supp([c_1 \wedge c_2]) \subset supp(c_1) \cap supp(c_2). \tag{4.1}$$

(2) (Cohomologicity) if c_1, c_2 are closed, $[c_1 \wedge c_2]$ is closed and

$$\langle [c_1 \wedge c_2] \rangle = \langle c_1 \rangle \smile \langle c_2 \rangle \tag{4.2}$$

where $\langle \bullet \rangle$ denotes the cohomology class of a singular cycle.

(3) (Leibniz rule) If $deg(c_1) = p$, then the differential map of chains follows Leibniz rule,

$$d[c_1 \wedge c_2] = [dc_1 \wedge c_2] + (-1)^p [c_1 \wedge dc_2], \tag{4.3}$$

where the differential map d is the operator $(-1)^{p+1} b$ for the boundary operator b acting on chains of the codimension p.

Proof. (1) Suppose

$$\mathbf{a} \notin supp(c_1) \cap supp(c_2).$$

Then $\mathbf{a}$ must be either outside of $supp(c_1)$ or outside of $supp(c_2)$. Let's assume first it is not in $supp(c_2)$. Since the support of a currents is closed, we choose a small neighborhood $U_{\mathbf{a}}$ of $\mathbf{a}$ in X, but disjoint from $supp(c_2)$. Let ϕ be a

C^∞-form of X with a compact support in $U_{\mathbf{a}}$. According to Theorem 2.2, when ϵ is small enough $R_\epsilon(c_2)$ is zero in $U_{\mathbf{a}}$. Hence

$$[c_1 \wedge c_2][\phi] = 0. \tag{4.4}$$

Hence $\mathbf{a} \notin supp([c_1 \wedge c_2])$. If $\mathbf{a} \notin supp(c_1)$, $U_{\mathbf{a}}$ can be chosen disjoint with $supp(c_1)$. Then since $\phi \in \mathscr{D}(U_{\mathbf{a}})$ is a C^∞-form of X with a compact support in $U_{\mathbf{a}}$ disjoint with $supp(c_1)$, the restriction of ϕ to c_1 is zero. Hence

$$[c_1 \wedge c_2][\phi] = 0.$$

Then $\mathbf{a} \notin supp([c_1 \wedge c_2])$. Thus

$$\mathbf{a} \notin supp(c_1) \cap supp(c_2)$$

will always imply

$$\mathbf{a} \notin supp([c_1 \wedge c_2]).$$

This completes the proof.

(2) By the homotopy formula (2.3), $R_\epsilon c_2$ is closed. Next let ϕ be a test form. By the definition

$$\begin{aligned} & b[c_1 \wedge c_2][\phi] \\ & = \lim_{\epsilon \to 0} \int_{c_1} R_\epsilon c_2 \wedge d\phi \\ & (\text{ since } c_1 \text{ is closed}) \\ & = \pm \lim_{\epsilon \to 0} \int_{c_1} dR_\epsilon c_2 \wedge \phi = 0. \end{aligned} \tag{4.5}$$

So $[c_1 \wedge c_2]$ is closed. For the closed test form ϕ, the supportive intersection number,

$$deg\Big(\big\langle [c_1 \wedge c_2] \big\rangle \smile \langle \phi \rangle\Big) \tag{4.6}$$

is a well-defined real number that is equal to

$$\lim_{\epsilon \to 0} c_1[R_\epsilon(c_2) \wedge \phi]. \tag{4.7}$$

By the de Rham theorem

$$c_1[R_\epsilon(c_2) \wedge \phi]$$

is the topological intersection number

$$\langle c_1 \rangle \smile \langle R_\epsilon c_2 \rangle \smile \langle \phi \rangle. \tag{4.8}$$

By Formula (2.3) again, $\langle R_\epsilon c_2 \rangle = \langle c_2 \rangle$. Thus

$$\lim_{\epsilon \to 0} c_1[R_\epsilon(c_2) \wedge \phi] = \langle c_1 \rangle \smile \langle c_2 \rangle \smile \langle \phi \rangle. \tag{4.9}$$

Formulas (4.9) and (4.6) imply

$$\langle [c_1 \wedge c_2] \rangle = \langle c_1 \rangle \smile \langle c_2 \rangle \tag{4.10}$$

(3) Let ϕ be a test form. Let

$$deg(c_1) = p, deg(c_2) = q.$$

Then

$$\begin{aligned}
&b[c_1 \wedge c_2][\phi] \\
&= \lim_{\epsilon \to 0} \int_{c_1} R_\epsilon c_2 \wedge d\phi \\
&\text{(Leibniz Rule for } C^\infty \text{ forms)} \\
&= \lim_{\epsilon \to 0} \int_{c_1} (-1)^q d(R_\epsilon c_2 \wedge \phi) + (-1)^{q+1} dR_\epsilon c_2 \wedge \phi \\
&= \lim_{\epsilon \to 0} (-1)^q \int_{bc_1} R_\epsilon c_2 \wedge \phi + \lim_{\epsilon \to 0} (-1)^{q+1} \int_{c_1} dR_\epsilon c_2 \wedge \phi \\
&\text{(By Formula (2.3), } d \text{ and } R_\epsilon \text{ commute)} \\
&= \lim_{\epsilon \to 0} (-1)^q \int_{bc_1} R_\epsilon c_2 \wedge \phi + \lim_{\epsilon \to 0} (-1)^{q+1} \int_{c_1} R_\epsilon dc_2 \wedge \phi \\
&= (-1)^q [bc_1 \wedge c_2][\phi] + (-1)^{q+1} [c_1 \wedge dc_2][\phi]
\end{aligned}$$

Hence

$$b[c_1 \wedge c_2] = (-1)^q [bc_1 \wedge c_2] + (-1)^{q+1} [c_1 \wedge dc_2]. \tag{4.11}$$

After change the sign, we found (4.11) is the same as (4.3).

□

Example 4.3. *Let $X = \mathbb{R}^2$. Let c_2 be the interval $(0, 1)$ on the x-axis and c_1 a curve diffeomorphic to c_1. We choose the local data which consists of the standard coordinates (x, y) for $\mathbb{R}^2$ and the bump function f satisfying*

$$\int_{\mathbb{R}^2} f(x, y) dx \wedge dy = 1 \tag{4.12}$$

Case 1: $c_2 = c_1$. Then $[c_1 \wedge c_2] = 0$.

Case 2: $c_1 \neq c_2$ such that c_1 meets c_2 at countably many points around the origin. Then the supportive intersection $[c_1 \wedge c_2]$ is a current supported at the intersection of sets. More precisely, let $\mathbb{R}^2 \times \mathbb{R}^2 = \Delta \times \Delta^\perp$ where Δ is *the diagonal and* $\Delta^{\perp}$ *is the orthogonal complement. Define the partial scalar multiplication* D_ϵ *as a linear map*

$$\begin{array}{ccc} \Delta \times \Delta^{\perp} & \to & \Delta \times \Delta^{\perp} \\ (\mathbf{a}, \mathbf{b}) & \to & (\epsilon \mathbf{a}, \mathbf{b}) \end{array} \tag{4.13}$$

Let $\pi : \Delta \times \Delta^{\perp} \to \Delta$ *be the orthogonal projection. Then there is measure* 0 *set* $S \subset \pi(c_1 \times c_2)$ *such that set-theoretically*

$$\lim_{\epsilon \to 0} D_{\frac{1}{\epsilon}} \left(\pi(c_1 \times c_2) \backslash S \right) \tag{4.14}$$

exists as a measurable set. Then for a test function ϕ *on* $\mathbb{R}^2$*,* $[c_1 \wedge c_2][\phi]$ *is a finite Lebesgue integral over the measurable set* $\lim_{\epsilon \to 0} D_{\frac{1}{\epsilon}} \left(\pi(c_1 \times c_2) \backslash S \right)$*. The intergrand is an* L^1 *function dependent of* f, ϕ*.*

The example shows the supportive intersection $[\bullet \wedge \bullet]$ depends on the regularization R_ϵ. But the following example shows that the situation could be otherwise.

Example 4.4. *We give two cases where the supportive intersections are independent of regularization* R_ϵ*. Both are classical and pave the way to the modern topological intersection number. Let* $X = \mathbb{R}^2$*. Let* f *be a bump function satisfying*

$$\int_{\mathbb{R}^2} f(x, y) dx \wedge dy = 1 \tag{4.15}$$

where x, y *are the coordinates of* $\mathbb{R}^2$*.*

Case 1: Let c_1 *be the interval* $(0, 1)$ *on the* x*-axis and* c_2 *a curve diffeomorphic to* c_1 *but crosses* x*-axis only once at the origin. Then*

$$[c_1 \wedge c_2] = \delta_{\mathbf{0}} \tag{4.16}$$

where $\delta_{\mathbf{0}}$ *is the* δ *function at the point* $\mathbf{0}$*. Formula (4.16) holds regardless how they meet tangentially.*

Case 2: Continue the setting in case 1. Let c_1 *be as above,* c_2 *a curve diffeomorphic to* c_1 *and meet* x*-axis only at the origin. But it does not cross the* x*-axis. Then*

$$[c_1 \wedge c_2] = 0. \tag{4.17}$$

Appendix A Orthogonal projection of a cell

Lemma A.1. *Let $p \leq m$ be two whole numbers. Let $\mathbb{R}^p, \mathbb{R}^{m-p}$ be subspaces of $\mathbb{R}^m$ such that $\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p}$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^p \times \{\mathbf{0}\}$ be the orthogonal projection. Let c be a p-dimensional regular cell in $\mathbb{R}^m$, ψ a smooth function on $\mathbb{R}^m$. Then there is a compactly supported L^1 function ξ_ψ on $\mathbb{R}^p \times \{\mathbf{0}\}$ such that*

$$\pi(T_c \wedge \psi) = \xi_\psi \tag{A.1}$$

where $\pi(current)$ denotes the pushforward on currents, and ξ_ψ represents a current of degree 0.

Proof. Let μ be the Lebesgue measure on $\mathbb{R}^p$, ϕ a test function. Let $C = c(\Pi_p)$. We should note that since T_c is a current with a compact support, the pushforward $\pi(T_c \wedge \psi)$ is a well-defined 0-current. Hence it is both a distribution and a 0-current. So it can be evaluated in two different ways, and the evaluation of the distribution $\pi(T_c \wedge \psi)$ at ϕ is equal to the current's evaluation at forms,

$$\pi(T_c \wedge \psi)[\phi d\mu] \tag{A.2}$$

which has the integral estimate

$$\begin{aligned} \Big|\pi(T_c \wedge \psi)[\phi d\mu]\Big| &\leq \left|\int_C \psi \wedge \pi^*(\phi) \wedge \pi^*(d\mu)\right| \\ &\leq M||\phi||_\infty \end{aligned} \tag{A.3}$$

where M is a constant independent of the test function and $||\bullet||_\infty = esssup|\bullet|$. Thus, $\pi(T_c \wedge \psi)$ as a distribution has order 0. Therefore it is a signed measure. Let $A \subset \mathbb{R}^p$ be a set of measure 0. Let $\overline{\pi} = \pi|_C$. So, $\overline{\pi}$ is a differential map between two manifolds of the same dimension p. Let

$$\overline{\pi}^{-1}(A) = E_1 + E_2$$

where E_1 is a set of critical points of $\overline{\pi}$, and $E_2 = \overline{\pi}^{-1}(A) \backslash E_1$. By the same estimate (A.3), we have

$$\Big|\pi(T_c \wedge \psi)[A]\Big| \leq M' \textstyle\int_{E_1+E_2} |d\mu| \tag{A.4}$$

where M' is a constant. By Sard's theorem $\int_{E_1} |d\mu| = 0$. We let $E_2 = \cup_{i=1}^\infty E_2^i$ such that

$$\overline{\pi}|_{E_2^i} : E_2^i \to \overline{\pi}(E_2^i) \tag{A.5}$$

is diffeomorphic. Then each $\overline{\pi}(E_2^i)$ is contained in A. Thus $\int_{E_2^i} |d\mu|$, which by the diffeomorphism (A.5) is the Lebesgue measure of $\overline{\pi}(E_2^i)$, must be 0. Hence

$$\Big|\pi(T_c \wedge \psi)[A]\Big| \leq \sum_{i=1}^\infty \int_{E_2^i} |d\mu| = 0.$$

Thus the signed measure $\pi(T_c \wedge \psi)$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^p$. By the Radon-Nikodym theorem ([1]), we obtain that the density function between the signed measure and the positive measure,

$$\xi_\psi = \frac{\pi(T_c \wedge \psi)}{\mu} \tag{A.6}$$

is an L^1 function. The numerator $\pi(T_c \wedge \psi)$ in the formula (A.6) indicates ξ_ψ has the bounded support $\pi(C)$. We complete the proof.

$\square$

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