On a Solution of the Inverse Spectral Problem for Differential Operators on a Finite Interval with Complex Weights

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Abstract. Non-self-adjoint second-order ordinary differential operators on a finite interval with complex weights are studied. Properties of spectral characteristics are established and the inverse problem of recovering operators from their spectral characteristics are investigated. For this class of nonlinear inverse problems an algorithm for constructing the global solution is obtained. To study this class of inverse problems, we develop ideas of the method of spectral mappings.

Keywords: differential operators, complex weight, spectral characteristics, inverse problem, method of spectral mappings

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1. Introduction

We consider the boundary value problem L for the differential equation

$$-y''(x) + q(x)y(x) = \lambda r(x)y(x), \quad 0 < x < T, \tag{1.1}$$

subject to the Robin boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(T) + Hy(T) = 0, \tag{1.2}$$

and the jump conditions at an interior point $b \in (0,T)$:

$$y(b+0) = d_1 y(b-0), \quad y'(b+0) = y(b-0)/d_1 + d_2 y(b-0). \tag{1.3}$$

Here λ is the spectral parameter, $q(x)$ and $r(x)$ are complex-valued functions, $q(x) \in L(0,T)$, and $r(x) = a_k^2$ for $x \in (b_{k-1}, b_k)$, where $0 = b_0 < b_1 = b < b_2 = T$. The numbers h, H, a_k and d_k are complex, and $a_k \neq 0, d_1 \neq 0$. For definiteness, let $\arg d_1 \in [0,\pi)$.

We study the inverse spectral problem for the boundary value problem (1.1)–(1.3). Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural science and technology. Inverse spectral problems are also used for solving nonlinear integrable evolution equations of mathematical physics. Inverse problems for the classical Sturm–Liouville operators (when $r(x) \equiv 1$, $d_1 = 1$, and $d_2 = 0$) have been studied fairly completely (see [1] and the historical review therein). Inverse problems for arbitrary order differential operators and systems with arbitrary characteristic numbers are more difficult. They have been solved later by the method of spectral mappings (see the monographs [2]–[3] and the references therein). Inverse problems on spatial networks are an important and popular part of the inverse problem theory; in the review paper [4] one can find the main results on inverse problems on spatial networks. Boundary value problems with discontinuous weights and jump conditions at interior points have been considered in many papers, but mostly for the case with real weights. In the case when $r(x) \equiv 1$ (i.e. $a_k = 1$), the boundary value problem L satisfying conditions (3) was studied in [5]–[9] and other papers. Inverse problems for a real weight $r(x)$ were studied in [10]–[14] and other works. Inverse problems for the boundary value problem L with complex-valued weights were studied in [15]-[16] where only uniqueness results were obtained. Note that complex-valued weights appear, in particular, in the study of the interaction of electromagnetic waves with layered media possessing both dielectric and magnetic properties [17]. Moreover, a number of problems for Sturm–Liouville equations on curves in the complex plane can be reduced to the boundary-value problem L of the form (1)–(3) on a real interval. In the present paper, we establish properties of the spectral characteristics for L, and study the inverse spectral problem of recovering parameters of L from the given spectral characteristics. For this class of nonlinear inverse problems an algorithm for constructing the global solution is obtained. To study this class of inverse problems, we develop ideas related to the method of spectral mappings [2].

2. Spectral data

Let $l_k := b_k - b_{k-1}$ and $a_k = r_k \exp(i\varphi_k)$, $r_k > 0$, $0 \le \varphi_2 < \varphi_1 < \pi$. We assume that the following regularity condition holds: $\omega_\pm := d_1 a_2 \pm a_1/d_1 \neq 0$. Denote by $\Phi(x,\lambda)$ the solution of (1.1) such that (1.3) holds and $U(\Phi) = 1$, $V(\Phi) = 0$. Let $M(\lambda) := \Phi(0,\lambda)$. We will also use the solutions $\varphi(x,\lambda), \psi(x,\lambda), S(x,\lambda)$ of Eq. (1.1) satisfying (1.3) and the conditions $\varphi(0,\lambda) = 1, \varphi'(0,\lambda) = h, S(0,\lambda) = 0, S'(0,\lambda) = 1, \psi(T,\lambda) = 1, \psi'(T,\lambda) = -H$. Denote $D(x,\lambda,\mu) := (\lambda-\mu)^{-1}\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle$, where $\langle y(x), z(x)\rangle := y(x)z'(x)-y'(x)z(x)$. The function

$$\Delta(\lambda) := \langle\varphi(x,\lambda),\psi(x,\lambda)\rangle = -V(\varphi) = U(\psi) \tag{2.1}$$

does not depend on x, and it is called the characteristic function for L. The eigenvalues $\Lambda := \{\lambda_k\}_{k\ge 0}$ of L coincide with the zeros of the entire function $\Delta(\lambda)$. Clearly,

$$\Phi(x,\lambda) = S(x,\lambda) + M(\lambda)\varphi(x,\lambda) = \psi(x,\lambda)/\Delta(\lambda),\ M(\lambda) = \Delta_0(\lambda)/\Delta(\lambda), \tag{2.2}$$

where $\Delta_0(\lambda) := \psi(0,\lambda) = V(S)$. Using (2.1) and (2.2) one gets

$$\langle\varphi(x,\lambda),\Phi(x,\lambda)\rangle \equiv 1. \tag{2.3}$$

Let $\lambda = \rho^2, \lambda_k = \rho_k^2$. Consider the half-planes $\Pi_k^\pm := \{\rho:\ \pm Im(\rho a_k) > 0\}$, $k = 1,2$, and denote

$$S_1 = \Pi_1^+ \cup \Pi_2^+,\ S_2 = \Pi_1^- \cup \Pi_2^+,\ S_3 = \Pi_1^- \cup \Pi_2^-,\ S_4 = \Pi_1^+ \cup \Pi_2^-.$$

Then $S_j = \{\rho:\ \arg\rho \in (\theta_j, \theta_{j+1})\}$, where $\theta_1 = \theta_5 = -\varphi_2, \theta_2 = \pi-\varphi_1, \theta_3 = \pi-\varphi_2, \theta_4 = -\varphi_1$. For sufficiently small $\delta > 0$ we construct the sectors $S_{j,\delta} := \{\rho:\ \arg\rho \in (\theta_j+\delta, \theta_{j+1}-\delta)\}$.

Let $\{e_k(x,\rho)\}_{k=1,2}$, $x \in [0,b]$ and $\{E_k(x,\rho)\}_{k=1,2}$, $x \in [b,T]$ be the Birkhoff-type fundamental systems of solutions (FSS's) of Eq. (1.1) with the asymptotics as $|\rho| \to \infty$, $\rho \in \overline{S_j}$, $\nu = 0,1$ (see [1]):

$$e_k^{(\nu)}(x,\rho) = ((-1)^{k-1} i\rho a_1)^\nu \exp((-1)^{k-1} i\rho a_1 x)[1], \quad x \in [0,b],$$

$$E_k^{(\nu)}(x,\rho) = ((-1)^{k-1} i\rho a_2)^\nu \exp((-1)^{k-1} i\rho a_2 (x-b))[1], \quad x \in [b,T].$$

where $[1] = 1 + O(1/\rho)$. The functions $e_k^{(\nu)}(x,\rho)$ and $E_k^{(\nu)}(x,\rho)$ are regular for $\rho \in S_j$, $|\rho| > \rho^*$ and continuous for $\rho \in \overline{S_j}$, $|\rho| \ge \rho^*$ for some $\rho^* > 0$. Using these FSS's and the jump conditions (1.3) we get the following asymptotical formulas as $|\rho| \to \infty$, $\nu = 0,1$:

$$\varphi^{(\nu)}(x,\lambda) = \Big((i\rho a_1)^\nu \exp(i\rho a_1 x)[1] + (-i\rho a_1)^\nu \exp(-i\rho a_1 x)[1]\Big)/2,\ x \in [0,b],$$

$$\varphi^{(\nu)}(x,\lambda) = \Big(\Big(\omega_+ \exp(i\rho a_1 l_1)[1] + \omega_- \exp(-i\rho a_1 l_1)[1]\Big)(i\rho a_2)^\nu \exp(i\rho a_2(x-b))[1] +$$

$$\Big(\omega_- \exp(i\rho a_1 l_1)[1] + \omega_+ \exp(-i\rho a_1 l_1)[1]\Big)(-i\rho a_2)^\nu \exp(-i\rho a_2(x-b))[1]\Big)/(4a_2),\ x \in [b,T],$$

$$\psi^{(\nu)}(x,\lambda) = \Big(\Big(\omega_+ \exp(i\rho a_2 l_2)[1] - \omega_- \exp(-i\rho a_2 l_2)[1]\Big)(i\rho a_1)^\nu \exp(i\rho a_1(b_1 - x))[1] +$$

$$\Big(-\omega_- \exp(i\rho a_2 l_2)[1] + \omega_+ \exp(-i\rho a_2 l_2)[1]\Big)(-i\rho a_1)^\nu \exp(-i\rho a_1(b_1 - x))[1]\Big)/(4a_1),\ x \in [0,b],$$

$$\psi^{(\nu)}(x,\lambda) = \Big((-i\rho a_2)^\nu \exp(i\rho a_2(T-x))[1] + (i\rho a_2)^\nu \exp(-i\rho a_2(T-x))[1]\Big)/2,\ x \in [b,T].$$

In view of (2.1), these formulas yield

$$\Delta(\lambda) = (-i\rho)\Big(\Big(\omega_+ \exp(i\rho a_1 l_1)[1] + \omega_- \exp(-i\rho a_1 l_1)[1]\Big)\exp(i\rho a_2 l_2)[1] -$$

$$\Big(\omega_- \exp(i\rho a_1 l_1)[1] + \omega_+ \exp(-i\rho a_1 l_1)[1]\Big)\exp(-i\rho a_2 l_2)[1]\Big)/4,\ |\rho| \to \infty, \tag{2.4}$$

$$M(\lambda) = \pm (i\rho a_1)^{-1}[1], \quad \rho \in \Pi_1^\pm. \tag{2.5}$$

Using (2.4) by the known technique (see [1, Ch.1]) we obtain that the spectrum Λ of L consists of two subsequences $\Lambda = \{\lambda_k\} = \{\lambda_{k1}\} \cup \{\lambda_{k2}\}$, and

$$\rho_{kj} = \sqrt{\lambda_{kj}} = \frac{k\pi}{r_j l_j}\exp(i\theta_{3-j}) + C_j + O(1/k),\ k \to \infty, \tag{2.6}$$

where $C_1 = -(2ia_1 l_1)^{-1}\ln(-\omega_-/\omega_+)$, $C_2 = (2ia_2 l_2)^{-1}\ln(\omega_+/\omega_-)$. Moreover,

$$|\Delta(\lambda)| \ge C|\rho \mathcal{E}_1(\rho l_1)\mathcal{E}_2(\rho l_2)|,\ |M(\lambda)| \le C/|\rho|,\ \lambda \in G_\delta, \tag{2.7}$$

$$|\varphi(x,\lambda)| \le C|\mathcal{E}_1(\rho x)|,\ x \in (0,b), \quad \forall\,\lambda,$$
$$|\varphi(x,\lambda)| \le C|\mathcal{E}_1(\rho x)\mathcal{E}_2(\rho(x-b))|,\ x \in (b,T), \quad \forall\,\lambda,$$
$$|\Phi(x,\lambda)| \le C|\rho\mathcal{E}_1(\rho x)|^{-1},\ x \in (0,b), \quad \lambda \in G_\delta,$$
$$|\Phi(x,\lambda)| \le C|\rho\mathcal{E}_1(\rho x)\mathcal{E}_2(\rho(x-b))|^{-1},\ x \in (b,T), \quad \lambda \in G_\delta,$$

where $G_\delta := \{\rho :\ |\rho - \rho_k|\} \ge \delta$, $\mathcal{E}_k(\rho x) := \exp(\pm i\rho a_k x)$ for $\rho \in \Pi_k^\pm$, $x \in l_k$. Let m_k be the multiplicity of the eigenvalue λ_k ($\lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+m_k-1}$), and put $S := \{k \ge 1 : \lambda_{k-1} \ne \lambda_k\} \cup \{0\}$. It follows from (2.6) that for sufficiently large k ($k > k^*$) all eigenvalues are simple, i.e. $m_k = 1$ for $k > k^*$. Similar to [18] one gets

$$M(\lambda) = \sum_{k \in S}\sum_{\nu=0}^{m_k-1}\frac{M_{k+\nu}}{(\lambda - \lambda_k)^{\nu+1}}\,, \tag{2.8}$$

where $\displaystyle\sum_\nu \frac{M_{k+\nu}}{(\lambda-\lambda_k)^{\nu+1}}$ is the principal part of $M(\lambda)$ in a neighborhood of λ_k. The sequence $\mathcal{M} = \{M_k\}_{k\ge 0}$ is called the Weyl sequence of L, and the data $W = \{\lambda_k, M_k\}_{k \ge 0}$ are called the spectral data of L. Similar to (2.6) we calculate $\mathcal{M} = \{M_{k1}\} \cup \{M_{k2}\}$, and

$$M_{k1} = \frac{2}{a_1^2 l_1}\Big(1 + O\Big(\frac{1}{k}\Big)\Big),\ M_{k2} = \frac{8}{\omega_-\omega_+ l_2}\exp\Big(\frac{2k\pi r_1 l_1}{r_2 l_2}(\cos\alpha + i\sin\alpha)\Big)\Big(1 + O\Big(\frac{1}{k}\Big)\Big), \tag{2.9}$$

as $k \to \infty$. Here $\alpha := \varphi_1 - \varphi_2 + \pi/2$. Note that $\cos\alpha < 0$, since $\alpha \in (\pi/2, 3\pi/2)$. Using (2.4), (2.6), (2.7), (2.9) and the asymptotical formulas for $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, we obtain the estimates

$$|\varphi(x,\lambda_{k1})| \le C, \quad |\varphi(x,\lambda_{k2})| \le C\exp\Big(\frac{-k\pi r_1 l_1\cos\alpha}{r_2 l_2}\Big), \quad x \in [0,T]. \tag{2.10}$$

It follows from (2.5) and (2.6) that

$$a_1 = \lim_{|\rho|\to\infty}(i\rho M(\lambda))^{-1}, \quad \rho \in \Pi_1^+, \tag{2.11}$$

$$l_1 = b = -\lim_{k\to\infty}(k\pi/(a_1\rho_{k1})), \quad l_2 = T - l_1, \quad a_2 = \lim_{k\to\infty}(k\pi/(l_2\rho_{k2}), \tag{2.12}$$

$$A := \omega_+/\omega_- = \lim_{k\to\infty}\exp(2i\rho_{k2}a_2l_2), \quad d_1 = \sqrt{(a_1(A+1))/(a_2(A-1))}. \tag{2.13}$$

3. Inverse problem

In this paper we consider the following inverse problem.

Inverse problem 3.1. *Given the Weyl function $M(\lambda)$ (or the spectral data W), construct L.*

According to (2.8) the specification of the Weyl function is equivalent to the specification of the spectral data.

Firstly, we will prove the uniqueness theorem. For this purpose, together with L we consider a boundary value problem $\tilde L$ of the same form but with $\tilde q(x), \tilde b, \tilde r(x), \tilde h, \tilde H, \tilde d_1, \tilde d_2$ instead of $q(x), b, r(x), h, H, d_1, d_2$. We agree that if a certain symbol χ denotes an object related to $L,$ then $\tilde\chi$ will denote an analogous object related to $\tilde L.$

Theorem 3.1. *If $M(\lambda)\equiv\tilde M(\lambda)$ (or $W=\tilde W$), then $L=\tilde L$. Thus, the specification of the Weyl function (or the spectral data) uniquely determines the functions $q(x), r(x)$ and the parameters b, h, H, d_1, d_2.*

Proof. It follows form (2.11)-(2.13) that $b=\tilde b,\ a_k=\tilde a_k,\ d_1=\tilde d_1$. We construct the functions

$$\mathcal{P}_0 = \Phi\tilde\varphi - \varphi\tilde\Phi, \quad \mathcal{P}_1 = \varphi\tilde\Phi' - \Phi\tilde\varphi'. \tag{3.1}$$

In view of (2.3), this yields

$$\varphi = \mathcal{P}_1\tilde\varphi + \mathcal{P}_0\tilde\varphi', \quad \Phi = \mathcal{P}_1\tilde\Phi + \mathcal{P}_0\tilde\Phi', \quad \mathcal{P}_1 - 1 = \varphi(\tilde\Phi' - \Phi') - \Phi(\tilde\varphi' - \varphi'). \tag{3.2}$$

Using (2.2), (3.1), (3.2) and the asymptotical formulas for φ and ψ, we infer

$$|\mathcal{P}_1(x,\lambda) - 1| \le C/|\rho|, \quad |\mathcal{P}_0(x,\lambda)| \le C/|\rho|, \quad \rho\in G_\delta\cap\tilde G_\delta. \tag{3.3}$$

Taking (2.2), (3.1) and the assumption of the theorem into account, we conclude that the functions $\mathcal{P}_k(x,\lambda)$ are entire in λ for each x. Together with (3.3) this yields $\mathcal{P}_1(x,\lambda)\equiv 1$, $\mathcal{P}_0(x,\lambda)\equiv 0$. Using (3.2) we calculate $\varphi(x,\lambda)\equiv\tilde\varphi(x,\lambda)$, $\Phi(x,\lambda)\equiv\tilde\Phi(x,\lambda)$, hence $L=\tilde L$. Theorem 3.1 is proved.

Let us go on to deriving a constructive solution of the inverse problem. For this purpose we will use ideas of the method of spectral mappings [2]. We will reduce our nonlinear inverse problem to the solution of the so-called *main equation*, which is a linear equation in a corresponding Banach space of sequences. We give a derivation of the main equation, and prove its unique solvability. Using the solution of the main equation we provide an algorithm for the solution of the inverse problem considered. For simplicity, in the sequel we confine ourselves to the case when the function $\Delta(\lambda)$ has only simple zeros (the general case requires minor technical modifications).

Let the Weyl function $M(\lambda)$ and the spectral data W be given. Using (2.15)-(2.17) we compute b, a_k and d_1. Then we choose a model boundary value problem $\tilde L$ such that $\tilde b = b, \tilde a_k = a_k, \tilde d_1 = d_1$, and arbitrary in the rest (for example, we can take $\tilde q = 0$). Let $\theta_k := 1$ if $\lambda_k = \lambda_{k1}$, and $\theta_k := \exp(-k\pi r_1 l_1 (r_2 l_2)^{-1}\cos\alpha)$ if $\lambda_k = \lambda_{k2}$. Denote

$$\xi_k := |\rho_k - \tilde\rho_k| + |M_k - \tilde M_k|\theta_k^2,\ z_{k0} := \lambda_k,\ z_{k1} := \tilde\lambda_k,\ \beta_{k0} := M_k,\ \beta_{k1} := \tilde M_k.$$

By virtue of (2.6) and (2.9) one has $\xi_k = O(1/k)$. Consider the functions

$$\varphi_{kj}(x) := \varphi(x, z_{kj}),\ \tilde\varphi_{kj}(x) := \tilde\varphi(x, z_{kj}),\ j = 0, 1,$$

$$B_{ni,kj}(x) := D(x, z_{ni}, z_{kj})\beta_{kj},\ \tilde{B}_{ni,kj}(x) := \tilde{D}(x, z_{ni}, z_{kj})\beta_{kj},\ i, j = 0, 1,$$
$$f_{k0}(x) := (\varphi_{k0}(x) - \varphi_{k1}(x))/(\xi_k\theta_k),\ f_{k1}(x) := \varphi_{k1}(x)/\theta_k,$$
$$A_{n0,k0}(x) := (B_{n0,k0}(x) - B_{n1,k0}(x))\xi_k\theta_k/(\xi_n\theta_n),$$
$$A_{n1,k1}(x) := (B_{n1,k0}(x) - B_{n1,k1}(x))\theta_k/\theta_n,\ A_{n1,k0}(x) := B_{n1,k0}(x)\xi_k\theta_k/\theta_n,$$
$$A_{n0,k1}(x) := (B_{n0,k0}(x) - B_{n1,k0}(x) - B_{n0,k1}(x) + B_{n1,k1}(x))\theta_k/(\xi_n\theta_n).$$

Similarly $\tilde{f}_{kj}(x)$ and $\tilde{A}_{ni,kj}(x)$ are defined. Using (2.6), (2.9), (2.10) and the asymptotical formulas for $\varphi(x, \lambda)$ we get

$$|f_{kj}(x)|,\ |\tilde{f}_{kj}(x)| \le C, \quad |A_{ni,kj}(x)|, |\tilde{A}_{ni,kj}(x)| \le C\xi_k(|n-k|+1)^{-1}. \tag{3.4}$$

Denote by V the set of indices $u = (n, i)$, where $n \ge 0$, $i = 0.1$.

Theorem 3.2. *The following relation holds*

$$\tilde{f}_{ni}(x) = f_{ni}(x) + \sum_{(k,j)\in V} \tilde{A}_{ni,kj}(x) f_{kj}(x),\ (n, i) \in V, \tag{3.5}$$

where the series converge absolutely and uniformly on $x \in [0, T]$ and λ on compact sets.

Proof. Consider the contours $\Gamma_N := \{\lambda : |\lambda| = R_N\}$, where $R_N \to \infty$ such that $\Gamma_N \subset G_\delta$. Denote $\mathcal{S}_k := \{\rho : Im(\rho a_k) = 0\}$, $\mathcal{S}_0 := \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S} := \{\rho : dist(\mathcal{S}_0, \rho) = \delta\}$, where $\delta > 0$ is such that $\Lambda \cup \tilde{\Lambda} \subset int\,\mathcal{S}$. Let γ be the image of $\mathcal{S}$ in the λ- plane, and $\Gamma'_N := \Gamma_N \cap int\,\gamma$, $\Gamma''_N := \Gamma_N \setminus \Gamma'_N$, $\gamma^*_N := \gamma \cap int\,\Gamma_N$. Denote by $\gamma_N := \gamma^*_N \cup \Gamma'_N$ and $\gamma^0_N := \gamma^*_N \cup \Gamma''_N$ the closed contours with counterclockwise circuit. Applying Cauchy's integral formula we get

$$\mathcal{P}_k(x, \lambda) - \delta_{1k} = \frac{1}{2\pi i}\int_{\gamma^0_N} \frac{\mathcal{P}_k(x, \mu) - \delta_{1k}}{\lambda - \mu} d\mu = \frac{1}{2\pi i}\int_{\gamma_N} \frac{\mathcal{P}_k(x, \mu)}{\lambda - \mu} d\mu - \frac{1}{2\pi i}\int_{\Gamma_N} \frac{\mathcal{P}_k(x, \mu) - \delta_{1k}}{\lambda - \mu} d\mu,$$

where $k = 0, 1$, $\lambda \in int\,\gamma^0_N$, and δ_{jk} is the Kronecker delta. Taking (3.2) into account we calculate

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i}\int_{\gamma_N} \Big(\tilde{\varphi}(x, \lambda)\mathcal{P}_1(x, \mu) + \tilde{\varphi}'(x, \lambda)\mathcal{P}_0(x, \mu)\Big)\frac{d\mu}{\lambda - \mu} + \varepsilon_N(x, \lambda).$$

In view of (3.3), one has $\lim\limits_{N\to\infty} \varepsilon_N(x, \lambda) = 0$ uniformly in $x \in [0, T]$ and λ on compact sets. Taking (3.1) and (2.2) into account we obtain

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \frac{1}{2\pi i}\int_{\gamma_N} \tilde{D}(x, \lambda, \mu)(M(\mu) - \tilde{M}(\mu))\varphi(x, \mu)d\mu + \varepsilon_N(x, \lambda).$$

Note that the terms with $S(x, \lambda)$ and $\tilde{S}(x, \lambda)$ are zero because of Cauchy's theorem. Using the residue theorem we get the relation

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{k=0}^{\infty}\Big(\tilde{B}_{ni,k0}(x)\varphi_{k0}(x) - \tilde{B}_{ni,k1}(x)\varphi_{k1}(x)\Big),$$

which is equivalent to (3.5). Theorem 3.2 is proved.

By similar arguments we calculate

$$A_{ni,kj}(x) - \tilde{A}_{ni,kj}(x) + \sum_{(l,s)\in V} \tilde{A}_{ni,ls}(x)A_{ls,kj}(x) = 0,\ (n, i), (k, j) \in V. \tag{3.6}$$

Let $f(x)=[f_u(x)]_{u\in V}$, $A(x)=[A_{u,v}(x)]_{u,v\in V}$, $\tilde f(x)=[\tilde f_u(x)]_{u\in V}$, $\tilde A(x)=[\tilde A_{u,v}(x)]_{u,v\in V}$. We denote by m the Banach space of bounded sequences $\chi=[\chi_u]_{u\in V}$ with the norm $\|\chi\|=\sup_{u\in V}|\chi_u|$. According to (3.4), one has that for each fixed x, the operators $I+\tilde A(x)$ and $I-A(x)$, acting from m to m, are linear bounded operators. Relations (3.5) and (3.6) can be written as follows

$$\tilde f(x)=(I+\tilde A(x))f(x),\quad (I+\tilde A(x))(I-A(x))=I.$$

Symmetrically one has $f(x)=(I-A(x))\tilde f(x)$, $(I-A(x))(I+\tilde A(x))=I$. Thus, for each fixed x, the operator $I+\tilde A(x)$ has a bounded inverse operator, hence the linear equation $\tilde f(x)=(I+\tilde A(x))f(x)$ is uniquely solvable. This equation is called *the main equation* of the inverse problem. Solving the main equation we find the vector $f(x)$, and also the solutions $\varphi_{ni}(x)=\varphi(x,\lambda_{ni})$ of Eq. (1.1), hence we can construct $q(x),h,H$ and d_2. Thus, the solution of the inverse problem can be found by the following algorithm.

Algorithm 3.1. *Given the Weyl function $M(\lambda)$ and the spectral data W.*
1) Calculate b,a_k and d_1 via (2.11)-(2.13).
2) Choose a model boundary value problem $\tilde L$ such that $\tilde b=b,\tilde a_k=a_k,\tilde d_1=d_1$.
3) Construct $\tilde f(x)$ and $\tilde A(x)$ (see above).
4) Find $f(x)=[f_u]_{u\in V}$ by solving the main equation $\tilde f(x)=(I+\tilde A(x))f(x)$.
5) Calculate $\varphi_{n1}(x)=f_{n1}(x)\theta_n,\varphi_{n0}=\varphi_{n1}(x)+f_{n0}(x)\xi_n\theta_n$.
6) Find $q(x),h,H$ and d_2 using (1.1)-(1.3).

Remark 3.1. We can also calculate $q(x)$ by the formula $q(x)=\tilde q(x)-2\mathcal{F}(x)$, where

$$\mathcal{F}(x)=\frac{d}{dx}\sum_{k=0}^{\infty}\Big(M_{k0}\tilde\varphi_{k0}(x)\varphi_{k0}(x)-M_{k1}\tilde\varphi_{k1}(x)\varphi_{k1}(x)\Big). \tag{3.7}$$

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