Derivation|Correction of Hardy-Littlewood Twin Prime Constant using Prime Generator Theory (PGT)

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February 5, 2024

Abstract

The Hardy–Littlewood twin prime constant is a metric to compute the distribution of twin primes. Using Prime Generator Theory (PGT) it is shown it is more easily mathematically and conceptually derived, and the correct value is a factor of 2 larger.

Introduction

In the early 1920's the British mathematicians, Godfrey Harold (G.H.) Hardy and John Edensor Littlewood teamed up to write (among a series of their many collaborations) on the topic of the Twin Primes problems. It will be shown their ***twin prime constant*** is off by a factor of 2, and its computation is conceptually simple, and mathematical easy to derive, using *Prime Generator Theory* (*PGT)*.

First Hardy–Littlewood Conjecture

The Hardy–Littlewood conjecture is a generalization of the Twin Prime conjecture. It's concerned with the distribution of *prime constellations*, including twin primes, in analogy to the *Prime Number Theory*.

A synopsis of its statement given by the Wiki2 article on *Polignac's Conjecture* [5] provides:

Conjectured Density

Let $\pi_n(x)$ (*n* even) be the number of prime gaps of size *n* below *x*.

The first Hardy–Littlewood conjecture says the asymptotic density is of form

(1) $$\pi_2(x) \sim 2C_2 \frac{x}{(\ln x)^2} \sim 2C_2 \int_2^x \frac{dt}{(\ln t)^2}$$

Here C_2, called the twin primes constant, is defined as follows:

(2) $$C_2 = \prod_{p\geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.66016181584686957392781211001 4\ldots$$

Polignac's Conjecture (1849) states – there are an infinite number of prime pairs that differ by any even value *n*. The *Twin Primes Conjecture* is for the specific case of prime gaps of 2, e.g. (5, 7) and (11, 13).

In a previous paper|video [1],[2] I established using *PGT* that *Polignac's Conjecture* is true. I'll first present some of its basic concepts, and its mathematical framework, that will be used to construct and explain the reasoning for the corrected *H-L* twin prime constant.

Modular Groups

Prime Generator Theory is derived from the properties of modular groups of size $\mathbb{Z}_n$, for even values *n*. When the number line is broken into successive groups of size *n*, the integers $\boldsymbol{r_i} < \boldsymbol{n}$ that are coprime to it are its ***residues***. They are the coprime primes (and possibly their multiples) not prime factors of *n*.

The number of residues of *n* are determined by the *Euler Totient Function (ETF)*, which has *PGT* form:

(3) $$\varphi(n) = n \prod_{i=1}^{j} \frac{(p_i - 1)}{p_i}$$

where the p_i are the *j* unique prime factors of *n*. For primorial values *n* = $p_m\#$ (the product of the first *m* primes), the residues count – ***rescntpn*** – for *n* = $p_m\#$ simply becomes:

(4) $$rescntpn = \prod_{i=1}^{m} (p_i - 1) = (p_m - 1)\#$$

The expression $(p_m - 1)\#$ (also written as $P_m^{-1}\#$) is the ***first reduced primorial*** of the first *m* primes. Reduced primorials have general form $(p_m - r)\#$ or $P_m^{-r}\#$ for $r \in \{1...p_m\}$, with $(0)\#$ = 1, for *r* = p_m. The principal and reduced primorials play a central role in the structure and the foundation of *PGT*.

Thus, *n* is the modulus – ***modpn*** – of $\mathbb{Z}_n$, of the integers {0...*n*-1}, whose residues count – ***rescntpn*** – is given by $\varphi(n)$, whose ***residues*** values are the coprime integers $\boldsymbol{r_i}$ to ***modpn***, e.g. *gcd*($\boldsymbol{r_i}$, ***modpn***) = 1.

Residue Gaps

As we create successively larger primorial modular groups, which contain more prime residues, the gap patterns between the residues completely characterize the gaps between the primes. The a_i residue gap coefficients hold the residue gaps counts, e.g. a_i is the number of residue gaps of size 2*i* for *n* = $p_m\#$. The table below shows for the first 11 primorial groups, their residue count $P_m^{-1}\#$, and their gap counts of sizes 2 and 4, i.e. a_1 and a_2, which are equal and odd, of the primorial the form: $a_{1|2} = (p_m - 2)\#$.

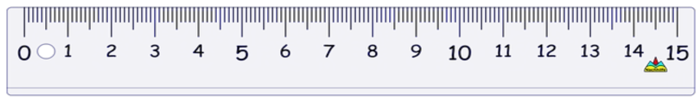
m	1	2	3	4	5	6	7	8	9	10	11
p_m	2	3	5	7	11	13	17	19	23	29	31
$p_m\#$	2	6	30	210	2,310	30,030	510,510	9,699,690	223,092,870	6,469,693,230	200,560,490,130
$(p_m - 1)\#$	1	2	8	48	480	5,760	92,160	1,658,880	36,495,360	1,021,870,080	30,656,102,400
$(p_m - 2)\#$	1	1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725	6,226,553,025

Figure 1.

From this simple deterministic mathematical framework, it is conceptually easy to see and understand, and mathematically derive, the physical modular group meaning of the twin prime constant. We will see it's the product of two ratios that define physical properties of primorial modular groups, whose correct value is a factor of 2 larger than that derived by Hardy-Littlewood.

Ruler Rules

A useful visual metaphor is to imagine the modular group size – ***modpn*** – as a ruler with integer units.



Rulers consists of gaps of a_i sizes and markers to designate their size. Here the little marks are all the group's integers and the big marks residues. The gaps and markings adhere to the 2 basic ***Ruler Rules***:

RR1: total length = sum of all gaps = Σgaps

RR2: number of residues = number of gaps = Σa_i

For *n* = $p_m\#$, the average gap size (integers per gap) is the length divided by the number of residues. The rulers are $P_m\#$ integers long with $P_m^{-1}\#$ residues|gaps, thus the average integers/residue-gap are:

(5) $$\hat{g}_n = \frac{P_m\#}{P_m^{-1}\#} = \frac{p_m\#}{(p_m-1)\#}$$

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\hat{g}_n$	2	3	3.75	4.375	4.8125	5.214	5.539	5.847	6.113	6.331	6.652	6.724	6.892	7.056

Figure 2.

Here, each larger p_m ratio is decreasing, approaching 1.0 from above, i.e. $p_m/(p_m-1) \rightarrow$ 1.00000... However, the product of their successive ratios grows (slowly) infinitely larger in value as $p_m \rightarrow \infty$.

Twins|Cousins Ratios

Twin and Cousin primes are prime pairs that differ by 2 and 4. The coefficients a_1 and a_2 indicate their pair candidates count within a modular group. Thus their primorial pairs count values – ***pairstcntpn*** – is given by $(p_m-2)\#$, which are the number of residue pair gaps of 2 and 4 for $\mathbb{Z}_n$, when *n* = $p_m\#$.

Thus, let – $\boldsymbol{r_2}$ – be the percentage of gap sizes of 2|4 among the total number of residue gaps, given by:

(6) $$r_2 = \frac{2|4 \ gaps \ count}{total \ gaps \ count} = \frac{(p_m-2)\#}{(p_m-1)\#}$$

Here, the ratios are increasing, approaching 1.0 from below, i.e. $(p_m-2)/(p_m-1) \rightarrow$ 0.99999.… with their successive ratio products becoming smaller $\rightarrow$ 0 with each additional prime.

PGT Derivation

From (1) notice the factor 2 in $2C_2$ is included, because *H-L* used only the odd primes, and excluded the first prime p_1 = 2 from the primes products values. This is probably because of the (p – 2) reduced prime factor in the numerator of (2), whose literal arithmetic computation for p = 2 makes everything zero. However, *PGT* informs us that conceptually the numerator|denominator multiplications in (2) are primorials, which represent physical modular group characteristics.

When p_1 = 2 is included in (2), the numerator|denominator expressions become primorials over all p:

(7) $$C_2 = \frac{p_m\#(p_m - 2)\#}{(p_m - 1)\#^2}$$

And for p_1 = 2 we get the added factor of 2 in (1).

(8) $$C_2 = \frac{2\#(0)\#}{1\#^2} = \frac{2(1)}{1} = 2$$

Thus C_2 as written can be conceptually understood as the ratio of:

(9) $$C_2 = \frac{modpn \cdot pairscntpn}{rescntpn^2}$$

But this expression can be broken into the product of 2 ratios of physical primorial group quantities.

(10) $r_1 = \dfrac{modpn}{rescntpn} = \dfrac{p_m\#}{(p_m - 1)\#}$ the (increasing) average modular residue gap size $\hat{g}_n$

(11) $r_2 = \dfrac{pairscntpn}{rescntpn} = \dfrac{(p_m - 2)\#}{(p_m - 1)\#}$ the (decreasing) percentage % of twin|cousin pair gaps

The conceptual and physical meaning of this is now easy to understand. As the residues|gaps increase as more primes are used to form the $\mathbb{Z}_n$ modular groups, the average integer distance|gaps between the residues increase, which for gaps of 2 and 4, is offset by a proportional countervailing decrease in their residue gaps percentage of the total number of gaps, such that their product is constant.

Thus r_1 will grow (slowly) without end, while r_2 decreases to near zero without end. Their product C_2, however, approaches an increasing stable equilibrium (more unchanging digits) as more primes are used to form the $\mathbb{Z}_n$ modular groups.

PGT informs us, as we use more primes to separate the number line into groups of n = $p_m\#$ integers, the prime residues are squeezed into a decreasing percentage of the integer number space they exist in. Thus the residue gaps increasingly become gaps strictly between primes. Because residue gaps of 2 and 4 are ***atomic***, and mathematically expressible as primorial values, their percentage is a constant ratio of all the residues gaps, and thus also for Twin and Cousin primes. And as there are an infinite number of primes, so too for any even gap size value between their pairs [1],[2].

This Ruby code computes r_1, r_2, C_2 using 1,000 primes, which can be increased for more stable digits.

```
require "primes/utils"         # external rubygem by Jabari Zakiya
                               # install first with $ gems install primes-utils
primes1000 = (1000.nthprime).primes # Array of first 1,000 primes (2..7919)

# Compute ratios: r1, r2, and c2, and print values for every 100th prime
r1, r2 = 2, 1                  # initialize values for first prime p=2

# Start using array primes from 2nd prime p=3, i.e. primes1000[1]
primes1000[1..].each_with_index do |p, i|
   p_1 = (p - 1.0)             # make p_1 a floating point value
   r1 *= p / p_1               # update  r1 for current prime
   r2 *= (p - 2) / p_1         # update  r2 for current prime
   c2  = r1 * r2               # compute c2 for current prime
   pth_prime = i + 2           # prmth val for current prime
   if pth_prime % 100 == 0     # if prmth value a multiple of 100 output results
      puts "\nUpto #{pth_prime}th prime #{p}"
      puts "r1 = #{r1} \nr2 = #{r2} \nC2 = #{c2} \n"
   end
end
```

This table shows the computed data produced by the code for the given number of primes shown.

m primes	r1	r2	C2
100	11.267620389582685	0.11720847212721400	1.3206605703724303
200	12.714305399732526	0.10385598009615430	1.3204566485310485
300	13.553362181029541	0.09742244361035730	1.3204016628121005
400	14.143788321617125	0.09335387299959709	1.3203774187094295
500	14.600162046719854	0.09043488777762677	1.3203640162302757
600	14.972683892976060	0.08818429692348541	1.3203556021596883
700	15.285994047978852	0.08637645148081055	1.3203499232212041
800	15.556739346163312	0.08487291685854093	1.3203458451169110
900	15.795529761036450	0.08358964822246509	1.3203427762125148
1000	16.008556779361957	0.08247716653126569	1.3203404034166584
500M	41.186500241230260	0.03205719407975786	1.3203236316991123

Figure 3.

The last row for 500M primes (up to 11,037,271,757) is shown to provide some perspective on growth. Below are the *H-L* and *PGT twin prime constant* values, and computed *PGT* value for 500M primes.

H-L: C2 = 0.660161815846869573927812110014...
PGT: C2 = 1.320323631693739147855624220028...
500M: C2 = 1.3203236316991123

Thus we see 500M primes give 12 accurate|stable digits, which will increase as more primes are used. Using languages|code that provide higher floating point precision can provide more accurate digits.

A plot of the log values of the data highlights better the slow rates of change for r_1 and r_2 , which is why after 500M primes C_2 only has 12 accurate|stable digits.

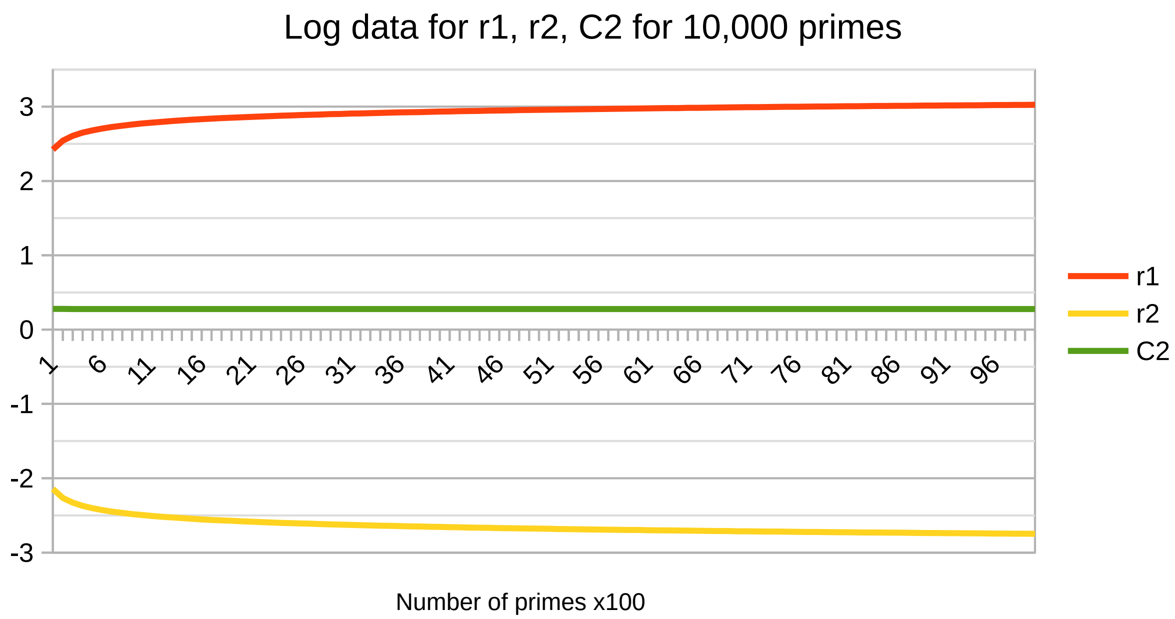


Figure 4.

Conclusion

When Hardy-Littlewood constructed their multiplicative form for the *twin prime constant* they used only the odd primes, excluding the first prime 2. This was likely because of the (*p* – 2) factor in the numerator, which they didn't know how to conceptually deal with. However, they realized they had to include a factor of 2 to the twin primes distribution integral to make the computations work.

Applying the conceptual understanding that *PGT* provides of its mathematical framework derived from modular groups, we see and understand that the external factor of 2 Hardy-Littlewood attached to their constant is actually a part of it, upon the realization that the prime multiplications can be conceptually treated, and computed, as multiplications of primorials.

We further see these primorial forms represent physical characteristics of modular groups $\mathbb{Z}_n$, when we break the number line into successive size groups of $n = p_m\#$ integers. Then, residue gaps of 2 and 4 completely characterize the distribution for Twin and Cousin primes (as well for the other residue gap size coefficients profiles do for the infinite prime pairs that differ by their values).

Thus, *PGT* provides a conceptually simple, yet mathematically powerful, framework to understand and characterize the distribution and relationship of primes, using only elementary concepts, arithmetic, and logic.

References

[1] On The Infinity of Twin Primes and other K-tuples; Jabari Zakiya, 2023;
https://vixra.org/pdf/2006.0053v3.pdf
https://www.academia.edu/41024027/On_The_Infinity_of_Twin_Primes_and_other_K_tuples

[2] (Simplest) Proof of Twin Primes and Polignac's Conjectures; Jabari Zakiya, 2021, video;
https://www.youtube.com/watch?v=HCUiPknHtfY

[3] Twin Primes Segmented Sieve of Zakiya (SSoZ) Explained; Jabari Zakiya, 2023;
J Curr Trends Comp Sci Res 2(2), 119 - 147, 2023; https://www.opastpublishers.com/open-access-articles/twin-primes-segmented-sieve-of-zakiya-ssoz-explained.pdf

[4] PRIMES-UTILS HANDBOOK; Jabari Zakiya, 2016;
https://www.academia.edu/19786419/PRIMES_UTILS_HANDBOOK

[5] Polignac's Conjecture; Wiki2, https://wiki2.org/en/Polignac%27s_conjecture