On Discrete Hopf fibrations, Grand Unification Groups, the Barnes-Wall, Leech Lattices, and Quasicrystals

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Abstract

A discrete Hopf fibration of S^{15} over S^8 with S^7 (unit octonions) as fibers leads to a $16D$ Polytope P_{16} with 4320 vertices obtained from the convex hull of the $16D$ Barnes-Wall lattice Λ_{16}. It is argued how a subsequent $2-1$ mapping (projection) of P_{16} onto a $8D$-hyperplane might furnish the 2160 vertices of the uniform 2_{41} polytope in 8-dimensions, and such that one can capture the chain sequence of polytopes $2_{41}, 2_{31}, 2_{21}, 2_{11}$ in $D = 8, 7, 6, 5$ dimensions, leading, respectively, to the sequence of Coxeter groups $E_8, E_7, E_6, SO(10)$ which are putative GUT group candidates. An embedding of the $E_8 \oplus E_8$ and $E_8 \oplus E_8 \oplus E_8$ lattice into the Barnes-Wall Λ_{16} and Leech Λ_{24} lattices, respectively, is explicitly shown. From the $16D$ lattice $E_8 \oplus E_8$ one can generate two separate families of Elser-Sloane $4D$ quasicrystals (QC's) with H_4 (icosahedral) symmetry via the "cut-and-project" method from $8D$ to $4D$ in each separate E_8 lattice. Therefore, one obtains in this fashion the Cartesian product of two Elser-Sloane QC's $\mathcal{Q} \times \mathcal{Q}$ spanning an $8D$ space. Similarly, from the $24D$ lattice $E_8 \oplus E_8 \oplus E_8$ one can generate the Cartesian product of three Elser-Sloane $4D$ quasicrystals (QC's) $\mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$ with H_4 symmetry and spanning a $12D$ space.

Keywords : Division Algebras, Hopf fibrations, Barnes-Wall lattice, Leech lattice, Exceptional Lie Algebras, Grand Unification, Quasicrystals.

1 Discrete Hopf Fibrations of S^{15} lead to the Polytopes associated with $E_8, E_7, E_6, SO(10)$

The four Hopf fibrations

$$S^1 \to S^1, \quad S^3 \to S^2, \quad S^7 \to S^4, \quad S^{15} \to S^8 \tag{1}$$

Dixon [1] discussed two specific Hopf lattice fibrations resulting from the *discrete* Hopf fibrations of S^7 over S^4, and S^{15} over S^8 [1]. One of them is the Hopf lattice fibration of the E_8 lattice over the Z^5 cross-polytope (with $2 \times 5 = 10$ vertices) where the fibers were provided by the 24 root vectors of the $D4$ lattice so that one generates the $10 \times 24 = 240$ roots of the E_8 lattice. Related to the last of the four Hopf fibrations, Dixon also discussed the Hopf lattice fibration of the 16-dim Barnes-Wall lattice Λ_{16} [2] over the cross-polytope (orthoplex) Z^9 with the E_8 lattice as fibers. The 240 root vectors of the E_8 lattice as fibers, and the cross-polytope (orthoplex) Z^9 as the base, with $2 \times 9 = 18$ vertices, leads to a total of $18 \times 240 = 4320$ lattice sites which matches the kissing number of the Λ_{16} Barnes-Wall lattice. Namely, the centers of the 4320 spheres packing the $16D$ space at each lattice site correspond to the 4320 vertices associated with the 4320 minimal vectors of the Λ_{16} lattice of norm 4.

It is well known (to the experts) that the 240 real roots of the E_8 Gossett 4_{21} polytope in $8D$ can be projected to *two* Golden-ratio scaled copies of the 120 root H_4 600-cell quaternion in $4D$, see [7] and references therein. The 600-cell in $4D$ has 120 vertices that correspond to the 120 roots of H_4. This very specific projection from $8D$ to $4D$ is possible due to the fact that the 8 simple roots of E_8 can be geometrically "folded" into two Golden-ratio scaled copies of the 4 simple roots of the Coxeter non-crystallographic group H_4 in 4-dim [7] ($240 = 2 \times 120$).

A convex polytope P_{16} in $16D$ can be geometrically obtained by taking the convex hull of the 4320 vertices associated to the 4320 minimal vectors of the Λ_{16} lattice. There is a uniform $8D$ polytope 2_{41} [8] with E_8 for its Coxeter group and which has 2160 vertices and $17520 = 240 + 17280$ **7**-faces. 240 of those **7**-faces are comprised of uniform 2_{31} polytopes with E_7 for their Coxeter group, and the other 17280 **7**-faces are **7**-simplices (higher dim version of the tetrahedron).

It is known that any finite simply-laced Coxeter-Dynkin diagram can be folded into $I_2(h)$ where h is the Coxeter number (height) which corresponds geometrically to the projection to the Coxeter plane. The number of roots is equal to the rank times the height. For example, in the case of E_8 one has $240 = 8 \times 30$, leading to 8 polygons with 30 vertices. Because none of the Coxeter groups in $16D$, $A_{16}, B_{16}, C_{16}, D_{16}$, can be geometrically "folded" into E_8, it is very unlikely that one will be able to project the P_{16} polytope to *two* Golden-ratio scaled copies of the uniform 2_{41} polytope in $8D$, and which would have been consistent with the $2160 + 2160$ splitting of the 4320 vertices of the parent $16D$ polytope P_{16}.

However, it is still plausible that the P_{16} polytope admits enough reflection symmetries such that one could find a judicious $8D$-hyperplane through the centroid of P_{16}, with the right orientation, and perform a $2-1$ map (projection) from $16D$ to $8D$ of all the 4320 vertices of P_{16}, and obtain the sought-after 2_{41} polytope with its 2160 vertices for the $8D$ projection. In other words, does the P_{16} polytope admit at least one $8D$ hyperplane for a "mirror" such that its 4320 vertices are symmetrically arranged into 2160 pairs with respect to this $8D$ "mirror" ?

In a given coordinate system, the 2160 vertices of the $8D$ polytope 2_{41} can be defined as follows [8] : there are 16 (2^4) vertices obtained from permutations of

$$(\pm 4, 0, 0^7) \tag{2}$$

where 0^7 denotes seven zero entries. There are 1120 ($16 \times C^8_4 = 16 \times 70$) vertices obtained from permutations of

$$(\pm 2, \pm 2, \pm 2, \pm 2, 0, 0, 0, 0) \tag{3}$$

and 1024 ($2^7 \times 8$) vertices of the form

$$(\pm 3, \pm 1, \pm 1, \ldots, \pm 1) \tag{4}$$

where the 1's must have an odd number of minus signs. The total number of vertices is 2160 and lie on a S^7 hyper-sphere of radius 4. In section **2** we shall explicitly display the coordinates of the 4320 minimal vectors of the Barnes-Wall lattice Λ_{16} of length-squared equal to 4 such that the tips of all the vectors (vertices) lie on a S^{15} hyper-sphere of radius 2. By joining the tips of all these vectors in S^{15} one constructs the convex polytope P_{16}. By a simple inspection, one finds that a *rescaling* of P_{16}, followed by an orthogonal projection to $8D$ will not generate the $2-1$ map yielding the 2160 vertices of 2_{41} displayed in eqs-(2,3,4).

However, this goal might be attained, firstly, by performing a rescaling of the vertices $\mathbf{V}$ of P_{16} : $\mathbf{V} \rightarrow \mathbf{V}' = \lambda \mathbf{V}$, with $\lambda > 1$, followed by a $SO(16)$ rotation of these rescaled vertices , $\mathbf{V}' \rightarrow \mathbf{V}''$, and a $SO(8)$ rotation of the vertices $\mathbf{W}$ of 2_{41} : $\mathbf{W} \rightarrow \mathbf{W}'$, and finally, one projects onto an $8D$ hyperplane the rescaled and rotated vertices of P_{16} . This projection π can can be realized in terms of a 8×16 *rectangular* matrix $\mathbf{M}$ that maps the 16 entries of $\mathbf{V}''$ into the 8 entries of $\mathbf{W}' \in 2'_{41}$. By a prime in $2'_{41}$ one means that the original polytope 2_{41} with coordinates given by eqs-(2,3,4) has been rotated. The $SO(16)$ rotations can be implemented via the use of the 120 bivectors Γ^{mn} of a Clifford algebra $Cl(16)$ in $16D$. While the $SO(8)$ rotations can be implemented via the use of the 28 bivectors γ^{ab} of a Clifford algebra $Cl(8)$ in $8D$. In doing so, one has

$$(\mathbf{V}'') \;=\; \lambda \left(e^{i\theta_{mn}\Gamma^{mn}} \; \mathbf{V} \; e^{-i\theta_{mn}\Gamma^{mn}} \right), \quad \mathbf{V} \in P_{16}, \quad m, n = 1, 2, \ldots, 16; \;\; \lambda > 1 \tag{5a}$$

where the Clifford vectors are $\mathbf{V} \equiv X_m \Gamma^m$, $\mathbf{V}'' \equiv X''_n \Gamma^n$. From eq-(5a) one can obtain the transformation of the coordinates $X''_n = X''_n(X_m)$. Because the

120 bivector Γ^{mn} generators do not commute (in general) one cannot factorize the exponential in eq-(5a) into a product of exponentials. The $SO(8)$ rotations involving the vertices $\mathbf{W}$ of 2_{41} are given by

$$(\mathbf{W}') = \left(e^{i\theta_{ab}\gamma^{ab}} \ \mathbf{W} \ e^{-i\theta_{ab}\gamma^{ab}} \right), \quad \mathbf{W} \in 2_{41}, \quad a, b = 1, 2, \ldots, 8 \qquad (5b)$$

with $\mathbf{W} \equiv X_a \gamma^a$, $\mathbf{W}' \equiv X'_b \gamma^\sigma$. There are 28 bivector generators in $8D$ and from (5b) one obtains the transformation of the coordinates $X'_b = X'_b(X_a)$.

Consequently, the combined rescaling-rotation-projections leads to equations of the form

$$\pi(\mathbf{V}'') = \lambda \ \pi \left(e^{i\theta_{mn}\Gamma^{mn}} \ \mathbf{V} \ e^{-i\theta_{mn}\Gamma^{mn}} \right) \ = \ \left(e^{i\theta_{ab}\gamma^{ab}} \ \mathbf{W} \ e^{-i\theta_{ab}\gamma^{ab}} \right) \ = \ \mathbf{W}' \qquad (6)$$

such that the end result is that pair of vertices $\mathbf{V}_1, \mathbf{V}_2 \in P_{16}$ are mapped to a single vertex $\mathbf{W}$ of the 2_{41} polytope. It is in this way how the $2-1$ map from P_{16} to the 2_{41} polytope could be constructed, if *possible*. At first sight, as one scans through all the $4320, 2160$ vertices of $P_{16}, 2_{41}$, respectively, one encounters an over-determined system of equations whose number is much larger compared to the $28 + 120 + 128 + 1 = 277$ parameters at our disposal. However one must not forget that *not* all of the equations are *independent* due to the very large number of symmetries.

There are 120 antisymmetric parameters θ^{mn} associated with the $SO(16)$ rotations implemented by the 120 bivectors Γ_{mn} of the Clifford algebra $Cl(16)$ in $16D$. There are $8 \times 16 = 128$ parameters associated with the 8×16 entries of the rectangular matrix $\mathbf{M}$ implementing the $16D \to 8D$ projection. The total number is $120 + 128 = 248$ which agrees with the dimension of the $\mathbf{e}_{8(8)}$ algebra comprised of 128 non-compact Y_α (spinorial) generators and 120 compact $X_{\mu\nu}$ generators. A chiral spinor $\mathbf{S}_+$ in $16D$ has 128 entries. The (anti) commutators are $[X_{\mu\nu}, X_{\rho\sigma}] = \eta_{\mu\sigma} X_{\nu\rho} \pm$ permutations. $[X_{\mu\nu}, Y_\alpha] \sim \Gamma_{\mu\nu\alpha}^{\ \ \ \ \beta} \ Y_\beta$, and $\{Y_\alpha, Y_\beta\} \sim \Gamma^{\mu\nu}_{\ \ \ \alpha\beta} \ X_{\mu\nu}$, with $\mu, \nu = 1, 2, \ldots, 16$, and $\alpha, \beta = 1, 2, \ldots, 128$.

The fact that 128 spinorial generators Y_α of the $\mathbf{e}_{8(8)}$ algebra are linked to the above construction of the $2-1$ map of P_{16} to 2_{41} might be related to the fact that the spin group is the *double* cover of the rotation group. This property of spinors was crucial in the construction of E_8 from a Clifford algebra in $3D$ by [11]. The H_3 Coxeter group in $3D$ admits a natural lift to H_4 in $4D$, by simply adding one node in the Coxeter diagram, and in turn, the H_4 can be geometrically "unfolded" into E_8 via the reverse mechanism explained earlier : the 8 simple roots of E_8 can be geometrically folded into two Golden-ratio scaled copies of the H_4 roots.

One may ask, why focus our attention to the 2_{41} polytope in $8D$ with 2160 vertices, half as many as the 4320 vertices of P_{16} ? One of the reasons why the 2_{41} polytope is important is because the *centroids* of 240 of its **7**-faces (comprised of uniform 2_{31} polytopes with E_7 for their Coxeter group) are precisely positioned at the 240 vertices of the Gosset 4_{21} polytope in $8D$. As its 240 vertices represent the root vectors of the simple Lie group E_8, this Gosset polytope is sometimes referred to as the E_8 root polytope.There are a total of $2^8 - 1 = 255$ uniform polytopes with E_8 symmetry in $8D$[1].

Another very important and salient feature is that there is a chain-sequence of three polytopes $2_{41}, 2_{31}, 2_{21}$ in $D = 8, 7, 6$ dimensions whose Coxeter groups are E_8, E_7, E_6, respectively. In particular, the 7-dim facets of 2_{41} contains 2_{31} polytopes (and **7**-simplices), and in turn, the 6-dim facets of 2_{31} contains 2_{21} polytopes (and **6**-simplices).

There is also the sequence of three polytopes $4_{21}, 3_{21}, 2_{21}$ in $D = 8, 7, 6$ dimensions whose Coxeter groups are E_8, E_7, E_6, respectively[2]. One can proceed further by noticing that the 6-dim 2_{21} polytope has for 5-facets : (i) 27 2_{11} polytopes (5-orthoplexes, cross polytopes) with D_5 as their Coxeter group, and (ii) 72 5-simplices with A_5 for their Coxeter group. Therefore, one may descend still further along the chain of polytopes $\ldots 2_{21} \rightarrow 2_{11}$ leading to $E_6 \rightarrow E_5 = D_5 = SO(10)$.

One can see that these chain-sequences of polytopes are very relevant in constructing extensions of the Standard Model of particle physics because the groups $E_8, E_7, E_6, SO(10)$ are among the many candidates to construct grand unified theories (GUT) [12], [13], [14] beyond those based on the groups $SU(5)$ and $SU(4) \times SU(2) \times SU(2)$ (Pati-Salam). From $SO(10)$ there are two natural branching routes to the standard model group $SO(10) \rightarrow SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$, and $SO(10) \rightarrow SU(4) \times SU(2) \times SU(2) \rightarrow SU(3) \times SU(2) \times U(1)$.

Another physical application is that there are polytopes whose number of vertices has a one-to-one correspondence with the number of fundamental particles associated to the GUT model one hopes to construct. For instance, Boya [15] found a natural correspondence among the vertices of the self-dual 24-cell (the octacube) in $4D$ and the particle content of the minimal supersymmetric standard model that requires 128 bosons and 128 fermions in two different sets, the ordinary particles and their supersymmetric partners.

To sum up : starting from the $16D$ Polytope P_{16} with 4320 vertices (obtained from the convex hull of the Barnes-Wall lattice Λ_{16}), we conjectured that a $2-1$ projection onto a judicious $8D$-hyperplane could exist, implementing the adequate reflection symmetry, in order to furnish the 2160 vertices of the uniform 2_{41} polytope in 8-dimensions, so that one can then capture the chain sequence of polytopes $2_{41}, 2_{31}, 2_{21}, 2_{11}$ in $D = 8, 7, 6, 5$ dimensions, leading, respectively, to the sequence of Coxeter groups $E_8, E_7, E_6, SO(10)$, and which are putative GUT group candidates. All these findings resulted from the *discrete* Hopf fibration of S^{15} over S^8 [1] with S^7 (unit octonions) as fibers. And, in doing so, we hope to answer Dixon's question of whether or not his construction of the Barnes-Wall lattice Λ_{16} has any physical applications [1].

[1] One may notice that 255 is the number of generators of the Clifford $Cl(8)$ algebra excluding the unit generator

[2] There is also the sequence of $1_{42}, 1_{32}, 1_{22}$ polytopes in $D = 8, 7, 6$ dimensions whose Coxeter groups are E_8, E_7, E_6, respectively

2 The Barnes-Wall, Leech Lattices and the Cartesian Products of Quasicrystals

The Barnes-Wall Lattice

The Barnes–Wall lattice Λ_{16} is the 16-dimensional positive-definite even integral lattice of discriminant 28 with no norm-2 vectors. It is the sublattice of the 24-dim Leech lattice fixed by a certain automorphism of order 2, and is analogous to the Coxeter–Todd lattice [2].

There are 480 vectors obtained from permutations of

$$\frac{1}{\sqrt{2}}\,(\pm 2, \pm 2, 0^{14}) \tag{7}$$

where 0^{14} denotes 14 consecutive zero entries. And 3840 vectors obtained from permutations of

$$\frac{1}{\sqrt{2}}\,(\pm 1, \pm 1, \pm 1, \ldots \pm 1, 0^{8}) \tag{8}$$

where 0^8 denotes 8 consecutive zero entries. All the minimal vectors have norm 4 (these vectors are not roots) where by norm one means the length *squared* of the vectors. It is worth pointing out an interesting numerical coincidence with these numbers of $\{480, 3840\}$ vectors. There are $480 = 2 \times 240$ octonionic multiplication tables and $3840 = 16 \times 240$ split-octonionic multiplication tables [1]. Adding the numbers of vectors yields $2 \times 240 \times (1+8) = 4320$. We shall see below that in the case of the $24D$ Leech lattice one has $3 \times 240 \times (1 + 16 + 16^2) = 196560$ minimal vectors of norm 4 (these vectors are not roots).

The E_8 lattice is constructed from 112 vectors ($\frac{2^2 \times 8 \times 7}{2} = 112$) obtained from permutations of

$$(\pm 1, \pm 1, 0^6) \tag{9}$$

after taking an arbitrary combination of signs and an arbitrary permutation of coordinates. And 128 vectors ($2^7 = 128$) obtained from permutations of

$$\frac{1}{2}\,(\pm 1, \pm 1, \pm 1, \ldots, \pm 1) \tag{10}$$

with the condition that one takes an even number of minus signs.[3]. All roots have norm 2. The E_8 lattice is related to 240 integral octonions [5].

The purpose now is to embed the rank-16 lattice $E_8 \oplus E_8$ directly into a rescaling of Λ_{16} and establish a one-to-one correspondence among the $480 = 240 + 240$ roots of $E_8 \oplus E_8$ with 480 of the rescaled 4320 minimal vectors of the Λ_{16} lattice. The 16-dim lattice $E_8 \oplus E_8$ was instrumental in the construction of

[3] The requirement of having an even number of minus signs reduces the number from 2^8 to 2^7

the $10D$ Heterotic string (there is also the 16-dim lattice $\Lambda(D_{16})$ corresponding to $SO(32)$). Firstly, one performs a rescaling of the vectors in eqs-(7,8) by a factor of $\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}}\ (\pm 2, \pm 2, 0^{14}) \rightarrow \frac{1}{2}\ (\pm 2, \pm 2, 0^{14})\ =\ (\pm 1, \pm 1, 0^{14}) \tag{11}$$

$$\frac{1}{\sqrt{2}}\ (\pm 1, \pm 1, \pm 1, \ldots \pm 1, 0^{8}) \rightarrow \frac{1}{2}\ (\pm 1, \pm 1, \pm 1, \ldots \pm 1, 0^{8}) \tag{12}$$

And then one embeds the vectors in $8D$ into $16D$ by arranging the 8 entries of the $8D$-vectors in the following two ways

$$(\pm 1, \pm 1, 0^6 | \mathbf{0^8}), \quad (\mathbf{0^8} | 0^6, \pm 1, \pm 1) \tag{13}$$

And

$$\frac{1}{2}\ (\pm 1, \pm 1, \pm 1, \ldots, \pm 1 | \mathbf{0^8}), \quad \frac{1}{2}\ (\mathbf{0^8} | \pm 1, \pm 1, \pm 1, \ldots, \pm 1) \tag{14}$$

where we indicate by $\mathbf{0^8}$ an array of 8 extra zeros separated from the slot of the initial 8 entries in order to perform the embedding. In this way the entries in eqs-(11,12) have the same structure as the entries in eqs-(13,14), and by direct inspection one can see that the entries (after permutations in the appropriate slot) of eq-(13) describe $112 + 112$ of the vectors of $E_8 \oplus E_8$, while the entries (with an even number of minus signs) of eq-(14) describe the other $128 + 128$ vectors of $E_8 \oplus E_8$, and such that 240 vectors of one copy of E_8 are orthogonal to the 240 vectors of the second copy of E_8. Therefore, in this straightforward way one has *embedded* the rank-16 lattice $E_8 \oplus E_8$ into a rescaling of the Λ_{16} lattice. The E_8 lattice provides the maximal packing of spheres in $8D$. The Leech yields the maximal packing in $24D$ [6]. For further details of the mathematics of E_8 see [4].

The Leech Lattice

The Leech lattice is an even unimodular lattice in 24-dimensional Euclidean space. The minimal vectors of the $24D$ Leech lattice Λ_{24} [2] consists of : (i) 97152 ($2^7 \times 759$) vectors obtained from permutations of

$$\frac{1}{\sqrt{2}}\ (\pm 1^8,\ 0^{16}) \tag{15}$$

and an even number of minus signs. (ii) 1104 ($2 \times 24 \times 23$) vectors obtained from permutations of

$$\frac{1}{\sqrt{2}}\ (\pm 2^2,\ 0^{22}) \tag{16}$$

and (iii) 98304 ($2^{12} \times 24$) vectors obtained from permutations of

$$\frac{1}{2\sqrt{2}}\ (\mp 3,\ \ \pm 1^{23}) \tag{17}$$

The total number of vectors is 196560 which is the kissing number of the Leech lattice. The vectors have norm 4.[4]

Because the Λ_{16} Barnes-Wall lattice is a sublattice of the 24-dim Leech lattice L_{24}, one can embed the rank-24 lattice $E_8 \oplus E_8 \oplus E_8$ into a rescaling of the Leech lattice by the same factor of $\frac{1}{\sqrt{2}}$. One now embeds the vectors in $8D$ into $24D$ by arranging the 8 entries of the $8D$-vectors in the following three ways (involving the cyclic permutations of slots)

$$(\pm 1, \pm 1, 0^6 | \mathbf{0^8} | \mathbf{0}^8), \quad (\mathbf{0^8} | \mathbf{0^8} | 0^6, \pm 1, \pm 1) \quad (\mathbf{0^8} | 0^6, \pm 1, \pm 1 | \mathbf{0^8}) \tag{18}$$

and

$$\frac{1}{2}\ (\pm 1, \pm 1, \pm 1, \ldots, \pm 1 | \mathbf{0^8} | \mathbf{0}^8), \quad \frac{1}{2}\ (\mathbf{0}^8 | \mathbf{0^8} | \pm 1, \pm 1, \pm 1, \ldots, \pm 1),$$

$$\frac{1}{2}\ (\mathbf{0^8} | \pm 1, \pm 1, \pm 1, \ldots, \pm 1 | \mathbf{0}^8) \tag{19}$$

A simple inspection of eqs-(18,19) and eqs-(15,16) shows that one has an embedding of the rank-24 lattice $E_8 \oplus E_8 \oplus E_8$ into a rescaled Leech lattice L_{24} by a factor of $\frac{1}{\sqrt{2}}$.

The Leech lattice was instrumental in the 24-dimensional orbifold compactification of the 26-dim bosonic string down to two dimensions. The automorphism group of the string twisted vertex operator algebra is the Monster group as shown by [19], and whose order is close to 10^{54}.

The 120 elements of the group of *icosians* [2] are provided by 120 *unit* quaternions whose coefficients are comprised of elements of the form $a + b\tau$ belonging to the Golden field $\mathbf{Q}[\tau]$, with a, b rationals and $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the Golden ratio, and $\sigma = \frac{1}{2}(1 - \sqrt{5}) = 1 - \tau = -\frac{1}{\tau}$ is its Galois conjugate. An example of an *icosian* is the following unit quaternion

$$\mathbf{q} = \frac{1}{2}(\tau e_1 + \sigma e_2 + e_3) \Leftrightarrow \frac{1}{2}(0, \tau, \sigma, 1) \ = \ \frac{1}{2}(0, \tau, 1 - \tau, 1) \ \Rightarrow \mathbf{q}\bar{\mathbf{q}} = 1 \tag{20}$$

where the *icosian* $\mathbf{x} = \alpha e_o + \beta e_1 + \gamma e_2 + \delta e_3$ is represented by $\mathbf{x} = (\alpha, \beta, \gamma, \delta)$, and each entry belongs to $\mathbf{Q}[\tau]$.

There are two norms for such vectors [2]. The quaternionic norm $QN(\mathbf{x}) = \mathbf{x}\bar{\mathbf{x}}$ which is a number of the form $u + v\sqrt{5}$, with u, v rational. And the Euclidean norm $EN(\mathbf{x}) = u + v$. With respect to the quaternionic norm the icosians belong to a four-dim space over the Golden field $\mathbf{Q}[\tau]$. But with respect to the Euclidean norm they lie in an eight-dim space. The latter Euclidean norm was instrumental in the Turyn-type construction for the Leech lattice based on the three-dim lattice over the icosians $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ [2].

Instead of using icosians to construct the Leech lattice, one can use octonions instead. To our knowledge the first one to use octonions in order to represent the Leech lattice over $\mathbf{O}^3$ was Dixon [1]. Wilson, later on [17] provided the following

[4]As a reminder, the norm of a vector is defined as the length squared

representation of the Leech lattice over $\mathbf{O}^3$: If L is the set of octonions with coordinates on the E_8 lattice, then the Leech lattice is the set of triplets (x, y, z) such that

$$x, y, z \in L; \quad x + y, \quad y + z, \quad x + z \in L\bar{s}; \quad x + y + z \in Ls \tag{21}$$

with

$$s \;=\; \frac{1}{2}\;(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \tag{22}$$

where $e_1, e_2, \ldots, e_7$ are the seven imaginary octonionic units squaring to -1.

The Dixon and Wilson's representations are actually equivalent as shown by [1]. The end result is that inner shell of Λ_{24} containing the minimal vectors is broken into three subsets with orders $3 \times 240; 3 \times 240 \times 16; 3 \times 240 \times 16^2$, respectively, the sum of all three orders being $3 \times 240 \times (1 + 16 + 16^2) = 196560$ which is the kissing number of the Leech lattice. The first subset with $3 \times 240 = 720$ vectors has a one-to-one correspondence with the 720 roots of the $E_8 \oplus E_8 \oplus E_8$ lattice as shown above corresponding to the canonical embedding of $E_8 \oplus E_8 \oplus E_8$ into a rescaling of Λ_{24} after a cyclic permutation of the entry slots as displayed by eqs-(18,19).

An intuitive explanation of the above $16, 16^2$ factors is the following. Since $24 = 8+16$, there are many ways to perform the embedding of an $8D$ basis frame of vectors into $24D$. The 240 roots of E_8 are given by linear combinations of the 8 *simple* roots $\beta_1, \beta_2, \ldots, \beta_8$ which comprise the $8D$ basis frame of vectors. There is room to perform translations of this $8D$ basis frame of vectors along the 16 *transverse* dimensions (to the 8 dimensions) in 24-dimensions. And also one can perform $GL(16, Z)$ "rotations" of this basis frame in the extra 16-dimensions. This simplistically explains the origins of the $16, 16^2$ factors in the above counting of minimal vectors. 16 for translations and 16×16 for $GL(16, Z)$ "rotations". The 16 discrete translations and $GL(16, Z)$ transformations can be combined into $GA(16, Z)$, the general affine group over the integers. There is still an extra factor of 3 (in 3×240) that escapes us but it might be related to the *triality* property of $SO(8)$.

Octonions and icosians can also be used to construct regular and uniform polytopes. The 600-cell in $4D$ has 120 vertices and H_4 is the Coxeter group. The coordinates of the locations of those 120 vertices in $4D$ can be represented in terms of the entries of 120 icosians (unit quaternions). Given the one-to-one correspondence between a vertex $\mathbf{V}$ and an icosian ι, one can define the group composition $\mathbf{V}_1 * \mathbf{V}_2$ of two vertices in terms of the quaternionic product of the two icosians as follows

$$\mathbf{V}_1 * \mathbf{V}_2 \;= \mathbf{V}_3 \;\Leftrightarrow\; \iota_1\; \iota_2 \;=\; \iota_3 \Leftrightarrow \mathbf{V}_3 \tag{23}$$

The upshot of establishing this vertex-icosian correspondence is that one can generate the positions of all the 120 vertices of the 600-cell from the composition law described by eq-(23) simply by starting with the quaternionic product of two icosians and generating the rest by successive iterations. An excellent video of the construction of the 120 vertices of the 600-cell based on the product of icosians can be found in [16].

The E_8 lattice [4] is also closely related to the nonassociative algebra of real octonions $\mathbf{O}$. It is possible to define the concept of an integral octonion analogous to that of an integral quaternion. The integral octonions naturally form a lattice inside $\mathbf{O}$ [1], [5]. This lattice is just a rescaled E_8 lattice. (The minimum norm in the integral octonion lattice is 1 rather than 2). Embedded in the octonions in this manner the E_8 lattice takes on the structure of a nonassociative ring [4].

A similar construction of the 120 vertices of the 600-cell in $4D$ works for the 240 vertices of the E_8 Gosset $8D$-polytope based on the integral octonions of norm 1. Because the octonions are a noncommutative and nonassociative normed division algebra, these 240 vertices have a multiplication operation which is *no* longer a group but rather a *loop*, in fact a Moufang loop [18]. In other words, the subset of unit-norm integral octonions is a finite Moufang loop of order 240, and which has a one-to-one correspondence with the 240 vertices of the E_8 Gosset polytope.

The octonions are nonassociative but alternative. On the other hand, the sedenions are not associative nor alternative, and are not a normed division algebra because they have 84 zero divisors[5]. As a result the norm of a product of two sedenions is not equal to the product of their norms. And because of this fact, it would be difficult to generate the coordinates of the locations of the vertices of polytopes in $16D$ from the products of unit sedenions.

We finalize this work with some remarks about lattices and Quasicrystals. From the $16D$ lattice $E_8 \oplus E_8$ one can generate *two* separate families of Elser-Sloane $4D$ quasicrystals (QC's) with H_4 (icosahedral) symmetry via the "cut-and-project" method from $8D$ to $4D$ in each separate E_8 lattice [9]. Therefore, one obtains in this fashion the Cartesian product of two Elser-Sloane QC's $\mathcal{Q} \times \mathcal{Q}$ spanning an $8D$ space. Because E_8 is a crystallographic group, and there are no non-crystallographic groups in $D > 4$, one cannot obtain an $8D$ QC via the "cut-and-project" method of the $16D$ Barnes-Wall Λ_{16} lattice down to an $8D$ model set. Instead one obtains the Cartesian product $\mathcal{Q} \times \mathcal{Q}$ of two $4D$ QC's with H_4 symmetry and spanning an $8D$ space. Similarly, from the $24D$ lattice $E_8 \oplus E_8 \oplus E_8$ one can generate the Cartesian product of three Elser-Sloane $4D$ quasicrystals (QC's) : $\mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$ with H_4 symmetry and spanning a $12D$ space.

A family of quasicrystals of dimensions $1, 2, 3, 4$ governed by the E_8 lattice was constructed by [10]. The *icosian* ring associated with the unit quaternions with coefficients in the Golden field $Q[\tau]$, and the standard "cut-and-projection" method from R^{2d} to R^d was instrumental in the construction. Nested sequences of quasicrystals formed systems whose symmetries were all derivable from the arithmetic of the *icosians*. The use of Coxeter diagrams clarified the relationship of E_8 and quasicrystal symmetries and lead to the fundamental chain $A_1 \times A_1 \subset A_4 \subset E_6 \subset E_8$ that underlies five-fold symmetry in quasicrystals. The role of the

[5] $84 = 14 \times 6$, where 14 is the dimension of the $\mathbf{g}_2$ algebra associated with G_2 which is the automorphism group of the octonions. And the factor of $6 = 3!$ corresponds to the order of the symmetric group S_3

non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$ in $D = 2, 3, 4$ dimensions, respectively, was essential.

Quasicrystalline compactifications of string theory based on a class of asymmetric orbifolds were constructed by [20]. The set of points of a one-dimensional cut-and-project quasicrystal or model set, while not additive, was shown to be multiplicative for appropriate choices of acceptance windows. This permits the introduction of Lie algebras over such aperiodic point sets [21]. More recently, (nonassociative) Jordan Algebras over Icosahedral cut-and-project QC have been constructed by [22].

The most immediate project is to test the existence of a $2 - 1$ map (projection) of P_{16} (with 4320 vertices) into a judicious $8D$ hyperplane leading to the 2_{41} polytope with 2160 vertices. If this is feasible one would have found a nice geometric framework of grand unified model groups, polytopes and discrete Hopf fibrations of (hyper) spheres which are deeply connected to the existence of the four normed division algebras : real, complex, quaternion and octonions [23]. Furthermore, it is worth exploring further the arguments of [24] related to how the ADE Coxeter graphs unify Mathematics and Physics.

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