Proof of the Riemann Hypothesis

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Abstract

This article shows that zeta function is a spiral on the complexe plane, then based on the general equation of the spiral we define an analytic continuation for zeta function and finally we prove that Reimann hypothesis is true.

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Summary

In this paper, we prove that the Riemann zeta function $\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s} : s=a+ib\,, s\neq 1$ on the complex plane is a spiral of radius r :

$$r = \lim_{N\to+\infty}\sqrt{\left(\frac{bN^{1-a}}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{A_{2j-1}}{N^{a+2j-1}}\right)\right)^2+\left(\frac{1}{2N^a}+\frac{(1-a)N^{1-a}}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{B_{2j-1}}{N^{a+2j-1}}\right)\right)^2}$$

With :

- $A_{2j-1}=\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)$
- $B_{2j-1}=\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)$
- $a\to K_n^p(a)$ $(a\in\mathbb{R})$ is the function that sums the multiplications between all the elements of the non-repeating combinations of the n elements $\{a; a+1; a+2; \dots; a+n-1 : n\in\mathbb{N}^*\}$ taken p by p $\left(C_n^p\right)$

And coordinate center :

$$\left(\left(\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}B_{2j-1}\right),\left(\frac{-b}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}A_{2j-1}\right)\right)$$

We then show that the Riemann zeta function can be extended analytically on all the complex plane except in $s=1$ by :

$$\boldsymbol{\zeta(s)}=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}B_{2j-1}\right]-\boldsymbol{i}\times\left[\frac{b}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}A_{2j-1}\right]$$

$$\equiv$$

$$\boldsymbol{\zeta(s)}=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i}\times\left[\frac{-b}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

Finally, we conclude that the Riemann hypothesis is true and that all non-trivial zeros of the zeta function have a real part $a = \frac{1}{2}$ and an imaginary part b that satisfies the equation :

$$\frac{1}{2} + \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} \left[K_{2j-1}^{2(j-n)-1}\left(\frac{1}{2}\right) - \frac{1}{2} K_{2j-1}^{2(j-n-1)}\left(\frac{1}{2}\right) \right] \right] = 0$$

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Introduction

The Riemann zeta function[1] , often referred to as $\zeta(s)$is a special mathematical function that plays an essential role in the study of the distribution of prime numbers and in number theory in general. It was introduced by the German mathematician Bernhard Riemann in the mid-19th century.

The Riemann zeta function is defined for complex numbers s of the form $s = a + bi$where *a* and *b* are real numbers, and *i* is the imaginary unit ($i^2 = -1$). The formula for the Riemann zeta function is as follows:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \sum_{n=1}^{+\infty} \frac{1}{n^{a+bi}}$$

When the real part of *s* is strictly greater than 1 ($a > 1$)the series converges, giving a finite value to the Riemann zeta function.

For $s = 1$ it has a simple pole and for any real part *s* strictly smaller than 1 ($a < 1$) the series diverges and can be analytically extended using the functional identity :

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s),$$

Where Γ is Euler's Gamma function. It then becomes possible to use this formula to define zeta for any negative real part *s* (with $\zeta(0) = -\frac{1}{2}$.

We deduce that strictly negative even integers are zeros of zeta (called trivial zeros). $\zeta(-2n) = 0$ and that non-trivial zeros are symmetrical about the axis $a = \frac{1}{2}$ axis and all have a real part between 0 and 1; this region of the complex plane is called the critical band.

As a result, Riemann's hypothesis can be reformulated as follows:

$$\zeta(s) = 0 \;\; Et \;\; 0 < a < 1\,, implique\ a = \frac{1}{2}$$

[1] Bernhard Riemann. Ueber die anzahl der primzahlen unter einer gegebenen grosse. Ges. Math. Werke und Wissenschaftlicher Nachlaß, 2(145-155):2, 1859

1 Literature review

1.1 Analytical extension principle

Theorem: Let U be a connected open of $\mathbb{C}$*let f* and g *be* two holomorphic functions on U, and let A be a part of U admitting an accumulation point that belongs to U. Then

$$f = g \text{ on } A \Leftrightarrow f = g \text{ on } U.$$

In particular, if $f = g$ in a neighborhood of a point a of U, then $f = g$ on U

This theorem is used to prove many uniqueness results for holomorphic functions. For example, the only holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ that verifies $f(\frac{1}{n}) = \frac{1}{n}$ for all $n \geq 1$ is the function $f(z) = z$. We apply the previous theorem to $A = \{\frac{1}{n} :\ n \geq 1\}$, $U = \mathbb{C}$noting that $0 \in \mathbb{C}$ is an accumulation point of A.

Definition[2] - Let U be an open of the set $\mathbb{C}$ of complex numbers and f an application of U in $\mathbb{C}$.

- We say that f is derivable (in the complex sense) or holomorphic at a point z_0 of U if the following limit, called the derivative of f at z_0 exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

- We say that f is holomorphic on U if it is holomorphic at any point in U.
- In particular, a holomorphic function on $\mathbb{C}$.

1.2 Bernoulli numbers

The Bernoulli numbers, noted B_n (or sometimes b_n so as not to confuse them with Bernoulli polynomials), are a series of rational numbers.

These numbers were first studied by Jacques Bernoulli[3] (which led Abraham de Moivre to give them the name we know today), looking for formulas to express sums of the type :

$$\sum_{k=0}^{n-1} k^m = 0^m + 1^m + 2^m + \cdots + (n-1)^m.$$

[2] https://www.bibmath.net/dico/index.php?action=affiche&quoi=./p/prolongementanalytique.html

[3] Jakob Bernoulli. Ars conjectandi: opus posthumum: accedit Tractatus de seriebus infinitis; et Epistola gallice scripta de ludo pilae reticularis. Impensis Thurnisiorum, 1713

For integer values of m, this sum is written as a polynomial of variable n whose first terms are :

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\left(n^{m+1} - \frac{1}{2}\binom{m+1}{1}n^m + \frac{1}{6}\binom{m+1}{2}n^{m-1} - \frac{1}{30}\binom{m+1}{4}n^{m-3} + \frac{1}{42}\binom{m+1}{6}n^{m-5} + \dots\right).$$

With : $$\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}.$$

The first Bernoulli numbers are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$

Jacques Bernoulli knew a few formulas like :

$$1+2+3+\cdots+(n-1) = \frac{1}{2}n^2 - \frac{n}{2} = \frac{n(n-1)}{2};$$

$$1^2+2^2+3^2+\cdots+(n-1)^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{n}{6} = \frac{n(n-1)(2n-1)}{6};$$

$$1^3+2^3+3^3+\cdots+(n-1)^3 = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{n^2(n-1)^2}{4};$$

$$1^4+2^4+3^4+\cdots+(n-1)^4 = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{n}{30} = \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30};$$

$$1^5+2^5+3^5+\cdots+(n-1)^5 = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 = \frac{n^2(n-1)^2(2n^2-2n-1)}{12}.$$

Bernoulli observed that the expression :

$$S_m(n) = \sum_{k=0}^{n-1} k^m = 0^m + 1^m + 2^m + \cdots + (n-1)^m$$

Is always a polynomial in n, of degree $m + 1$, and defines the Bernoulli numbers B_k by :

$$S_m(n) = \sum_{k=0}^{m} \frac{m!}{(m+1-k)!}\frac{B_k}{k!}n^{m+1-k} = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\,n^{m+1-k} = \sum_{k=0}^{m}\binom{m}{k}B_k\frac{n^{m+1-k}}{m+1-k}.$$

In particular, the coefficient of n in the polynomial $S_m(n)$ is the number B_k.

1.3 Special values of the Riemann zeta function[4]

1.3.1 In 0 and 1

In zero, we have : $\zeta(0) = -\frac{1}{2}$

In 1 there is a pole, so $\zeta(1)$ is not finite but the limit is $-\infty$on the left and $+\infty$right :

$$\lim_{s \to 1^{\pm}} \zeta(s) = \pm\infty$$

1.3.2 Positive even integers

The exact values of the zeta function at even positive integers can be expressed from Bernoulli numbers:

$$\forall n \in \mathbb{N},\ \zeta(2n) = (-1)^{n+1} \frac{(2)^{2n-1} B_{2n}}{(2n)!} \pi^{2n}.$$

1.3.3 Odd positive integers

There is no general formula for calculating the zeta function for odd positive integers.

The sum of the harmonic series is infinite:

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots = \infty$$

The value $\zeta(3)$ is also known as Apéry's constant (1.202..) and appears in the electron's gyromagnetic ratio. The value $\zeta(5)$ appears in Planck's law (1.036...).

1.3.4 Negative integers

In general, for any negative integer, we have :

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

Trivial" zeros are negative even integers:

$$\zeta(-2n) = 0 \quad (B_{2n+1} = 0 : n > 0)$$

[4] https://fr.wikipedia.org/wiki/Valeurs_particuli%C3%A8res_de_la_fonction_z%C3%AAta_de_Riemann

2 Graphic observations

We introduce the function $s \rightarrow Z(s)$ for all s in the set of complex numbers $s = a + bi$such that :

$$Z(s) = \sum_{n=1}^{N} \frac{1}{n^s} = \sum_{n=1}^{N} \frac{1}{n^{a+bi}}$$

We have the following equality:

$$\lim_{N \to +\infty} Z(s) = \zeta(s)$$

And since :

$$\frac{1}{n^{a+bi}} = n^{-a} \times e^{-ibln(n)} = \frac{\cos(bln(n))}{n^a} - i \times \frac{\sin(bln(n))}{n^a}$$

This gives :

$$\zeta(s) = \lim_{N \to +\infty} \sum_{n=1}^{N} \frac{\cos(bln(n))}{n^a} - i \times \lim_{N \to +\infty} \sum_{n=1}^{N} \frac{\sin(bln(n))}{n^a} \qquad (1)$$

Figure 1 shows the graphical representation of $Z(s)$ in the complex plane when N varies between $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ for some real numbers (a, b).

Indications :

We know that :

- $\zeta\left(\frac{1}{2} \pm \text{ i} \times 49.773832478 \ldots\right) = 0$
- $\zeta\left(\frac{1}{2} \pm \text{ i} \times 101.317851006 \ldots\right) = 0$

49.773832478 ...; 101.317851006 ...are estimates of the imaginary part of some non-trivial zeros of the zeta function.

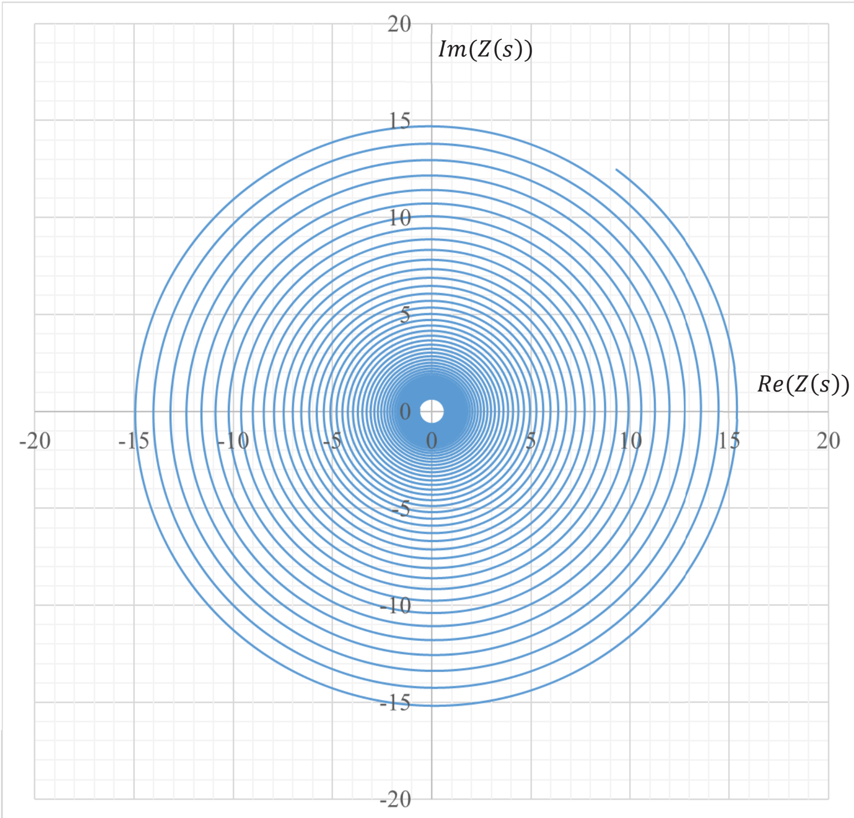

Figure 1 Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 49.773832478 \ldots$

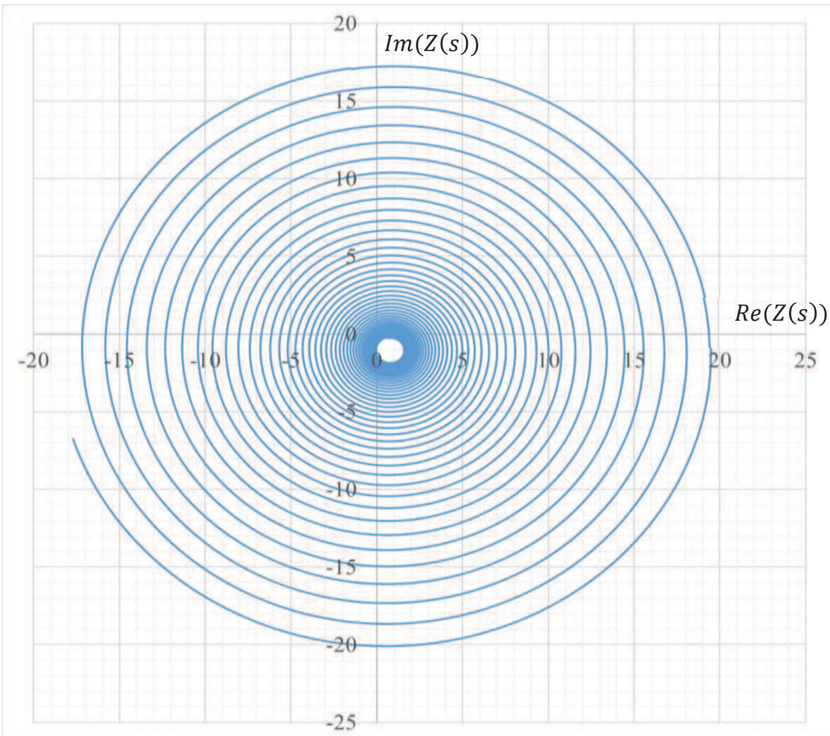

Figure 2: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 40$

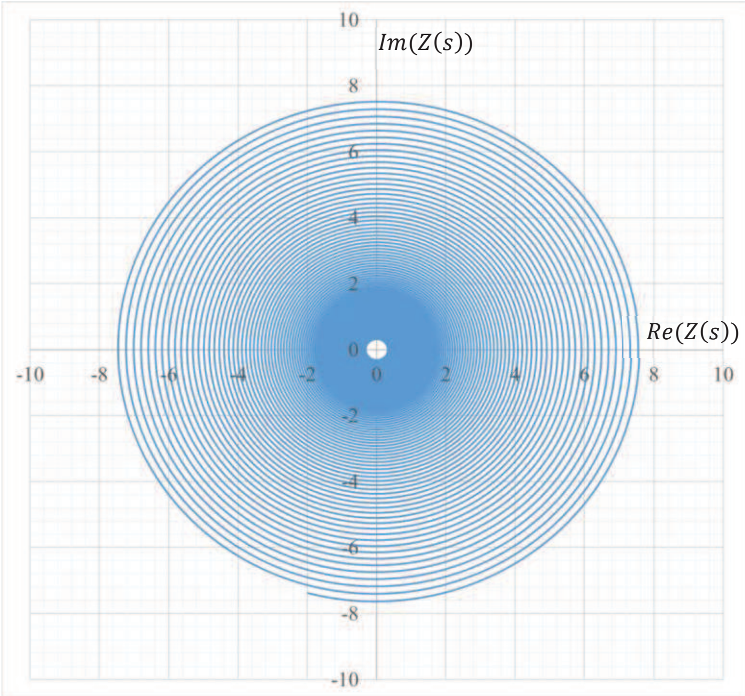

Figure 3: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 101.317851006 \ldots$

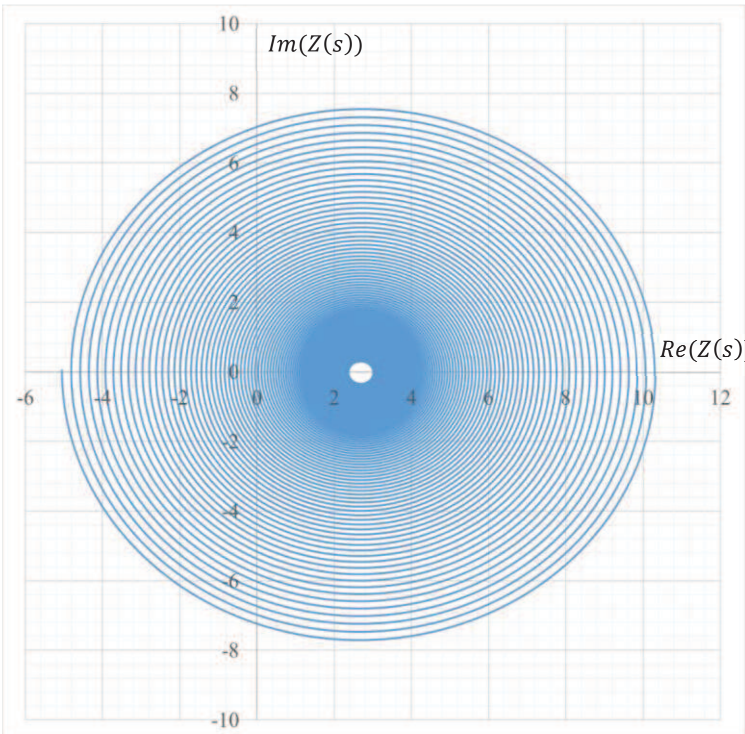

Figure 4: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 100$

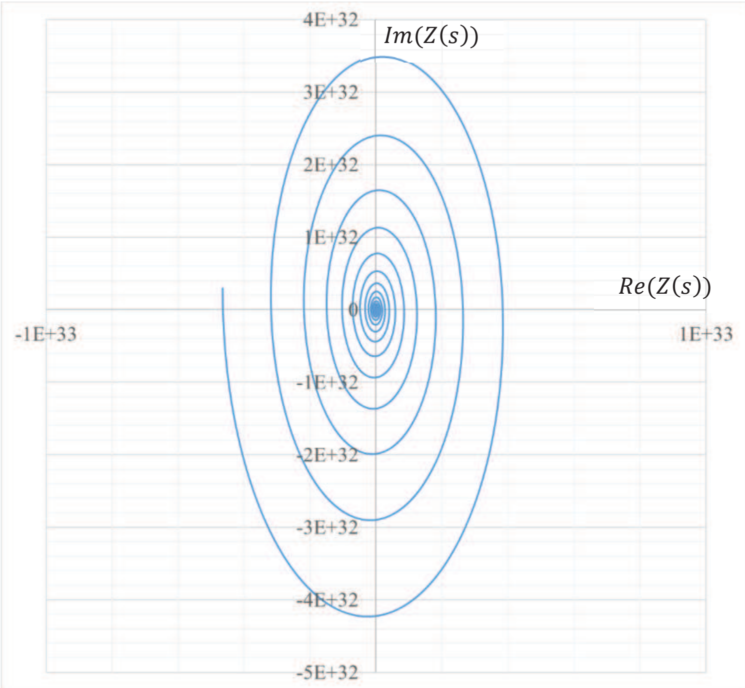

Figure 5: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = -5 + i \times 100$

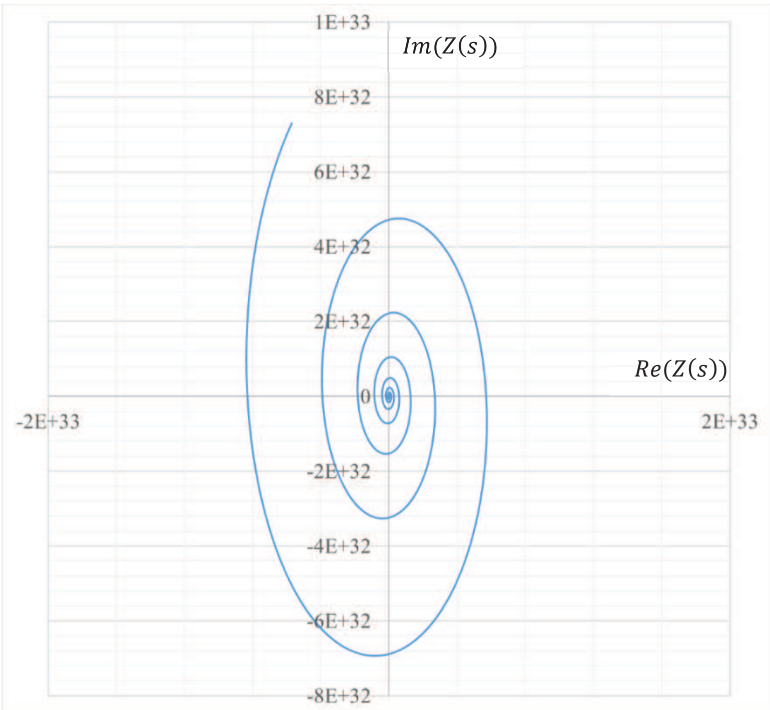

Figure 6: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = -5 + i \times 50$

From Figures 1, 2, 3, 4, 5 and 6, we can draw the following conclusions:

- When $N \to +\infty$, $Z(s) : s = a + bi$in the complex plane is a spiral with radius $r \in \mathbb{R}$ depending on *N*, a and b and a center with coordinates $(u, v) \in \mathbb{R}^2$ depending on a and b ;
- When $a = \frac{1}{2}$ and *b* takes one of the values of the imaginary part of the non-trivial zeros of the zeta function, the spiral appears to have the origin of the reference frame as its center (0,0).

3 Hypothesis (0)

Based on graphical observation, hypothesis (0) can be formulated:

The analytical extension of the Riemann zeta function can be written as $\zeta(s) = u + iv$ and *(u,v)* are the coordinates of the center of the *Z(s)* spiral towards infinity.

4 Demonstration

According to relationship (1) :

$$\zeta(s) = \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\cos(bln(n))}{n^a} - i \times \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\sin(bln(n))}{n^a}$$

Posing $f(n) = \frac{\cos(bln(n))}{n^a}$ and $g(n) = \frac{\sin(bln(n))}{n^a}$

By applying the Euler-Maclaurin formula, which can be expressed as follows:

For a function *f* continuously differentiable *2k* times on the segment [*p*, *q*] (with $k \geq 1$) we have :

$$\sum_{n=1}^{N} f(n) = \frac{f(1) + f(N)}{2} + \int_1^N f(x)dx + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(1)\right) + R_k$$

The numbers b_{2j} denote Bernoulli numbers, and the remainder R_k is expressed using the Bernoulli polynomial B_{2k} :

$$R_k = -\frac{1}{(2k)!} \int_p^q f^{(2k)}(x) B_{2k}(x - \lfloor x \rfloor)\, \mathrm{d}x.$$

The functions *f* and *g* are continuously derivable *2k* times on the segment [*1*, *N*] (with $k \geq 1$), then :

$$\sum_{n=1}^{N} f(n) = \frac{f(1)+f(N)}{2} + \int_1^N f(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(f^{(2j-1)}(N) - f^{(2j-1)}(1))$$

$$\sum_{n=1}^{N} g(n) = \frac{g(1)+g(N)}{2} + \int_1^N g(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(g^{(2j-1)}(N) - g^{(2j-1)}(1))$$

$$\lim_{k\to+\infty} R_k = 0$$

Calculation of $\int_1^N f(x)dx$ et $\int_1^N g(x)dx$:

$$\int_1^N \frac{\cos(bln(x))}{x^a} = \int_1^N \left[\frac{x^{1-a}}{1-a}\right]' \cos(bln(x)) = \left[\frac{x^{1-a}}{1-a}\cos(bln(x))\right]_1^N + \frac{b}{1-a}\int_1^N \frac{\sin(bln(x))}{x^a}$$

$$= \frac{N^{1-a}}{1-a}\cos(bln(N)) - \frac{1}{1-a} + \frac{b}{1-a}\int_1^N \frac{\sin(bln(x))}{x^a} \qquad (a \neq 1)$$

$$\int_1^N \frac{\sin(bln(x))}{x^a} = \int_1^N \left[\frac{x^{1-a}}{1-a}\right]' \sin(bln(x)) = \left[\frac{x^{1-a}}{1-a}\sin(bln(x))\right]_1^N - \frac{b}{1-a}\int_1^N \frac{\cos(bln(x))}{x^a}$$

$$= \frac{N^{1-a}}{1-a}\sin(bln(N)) - \frac{b}{1-a}\int_1^N \frac{\cos(bln(x))}{x^a}$$

Substitute $\int_1^N \frac{\cos(bln(x))}{x^a}$ and $\int_1^N \frac{\sin(bln(x))}{x^a}$ in both ties :

$$\int_1^N \frac{\cos(bln(x))}{x^a} = \frac{N^{1-a}}{1-a}\cos(bln(N)) - \frac{1}{1-a} + \frac{b}{1-a} \times \left[\frac{N^{1-a}}{1-a}\sin(bln(N)) - \frac{b}{1-a}\int_1^N \frac{\cos(bln(x))}{x^a}\right]$$

$$\equiv \left[1 + \left(\frac{b}{1-a}\right)^2\right] \times \int_1^N \frac{\cos(bln(x))}{x^a} = \frac{N^{1-a}}{1-a} \times \left[\cos(bln(N)) + \frac{b}{1-a}\sin(bln(N))\right] - \frac{1}{1-a}$$

$$\equiv \int_1^N \frac{\cos(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a} \times \left[\cos(bln(N)) + \frac{b}{1-a}\sin(bln(N))\right] - \frac{1}{1-a}\right]}{\left[1 + \left(\frac{b}{1-a}\right)^2\right]}$$

$$\int_1^N \frac{\sin(bln(x))}{x^a} = \frac{N^{1-a}}{1-a}\sin(bln(N)) - \frac{b}{1-a} \times \left[\frac{N^{1-a}}{1-a}\cos(bln(N)) - \frac{1}{1-a} + \frac{b}{1-a}\int_1^N \frac{\sin(bln(x))}{x^a}\right]$$

$$\equiv \left[1+\left(\frac{b}{1-a}\right)^2\right] \times \int_1^N \frac{\sin(bln(x))}{x^a} = \frac{N^{1-a}}{1-a} \times \left[\sin(bln(N)) - \frac{b}{1-a}\sin(bln(N))\right] + \frac{b}{(1-a)^2}$$

$$\equiv \int_1^N \frac{\sin(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a} \times \left[\sin(bln(N)) - \frac{b}{1-a}\sin(bln(N))\right] + \frac{b}{(1-a)^2}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

This gives :

$$\int_1^N \boldsymbol{f(x)dx} = \int_1^N \frac{\cos(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a} \times \left[\cos(bln(N)) + \frac{b}{1-a}\sin(bln(N))\right] - \frac{1}{1-a}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

$$\int_1^N \boldsymbol{g(x)dx} = \int_1^N \frac{\sin(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a} \times \left[\sin(bln(N)) - \frac{b}{1-a}\sin(bln(N))\right] + \frac{b}{(1-a)^2}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} \qquad (2)$$

Calculation of $\boldsymbol{f^{(2j-1)}(N)}$ ***et*** $\boldsymbol{f^{(2j-1)}(1)}$**:**

Calculating the first and second derivatives of *f* :

$$\left[\frac{\cos(bln(x))}{x^a}\right]' = \frac{-ax^{a-1}\cos(bln(x)) - bx^{a-1}\sin(bln(x)}{x^{2a}} = \frac{-a\cos(bln(x)) - b\sin(bln(x))}{x^{a+1}}$$

$$\left[\frac{\cos(bln(x))}{x^a}\right]'' = \left[\frac{-a\cos(bln(x)) - b\sin(bln(x))}{x^{a+1}}\right]' = -a \times \left[\frac{\cos(bln(x))}{x^{a+1}}\right]' - b \times \left[\frac{\sin(bln(x))}{x^{a+1}}\right]'$$

$$= -a \times \left[\frac{-(a+1)\cos(bln(x)) - b\sin(bln(x))}{x^{a+2}}\right] - b \times \left[\frac{b\cos(bln(x)) - (a+1)\sin(bln(x))}{x^{a+2}}\right]$$

$$= \frac{(a(a+1) - b^2)\cos(bln(x)) + (b(a+1) + ab)\sin(bln(x))}{x^{a+2}}$$

We can see that the derivative of order *k* of *f* can be written as :

$$f^{(k)} = \frac{A_k \sin(bln(x)) + B_k \cos(bln(x))}{x^{a+k}}$$

A_k *et* B_k Are factors that depend on *a* and *b* and the derivation order *k*.

$$A_1 = -b \quad B_1 = -a$$

$$A_2 = b(a+a+1) \quad B_2 = a(a+1) - b^2$$

$$f^{(k+1)} = \left[\frac{A_k \sin(bln(x)) + B_k \cos(bln(x))}{x^{a+k}}\right]' = A_k \times \left[\frac{\sin(bln(x))}{x^{a+k}}\right]' + B_k \times \left[\frac{\cos(bln(x))}{x^{a+k}}\right]'$$

$$= A_k \times \left[\frac{bcos\big(bln(x)\big)-(a+k)\sin\big(bln(x)\big)}{x^{a+k+1}}\right] + B_k \times \left[\frac{-bsin\big(bln(x)\big)-(a+k)\cos\big(bln(x)\big)}{x^{a+k+1}}\right]$$

$$= \frac{(A_k\times b-B_k\times(a+k))\cos(bln(x))-(A_k\times(a+k)+B_k\times b)\sin(bln(x))}{x^{a+k+1}}$$

And so we have the following two equalities:

$$A_{k+1} = -(A_k \times (a+k) + B_k \times b)$$

$$B_{k+1} = (A_k \times b - B_k \times (a+k)) \qquad (3)$$

They are used to calculate the rest of the factors:

$$A_3 = -(A_2 \times (a+2) + B_2 \times b)$$

$$= -(b(a+a+1) \times (a+2) + (a(a+1) - b^2) \times b)$$

$$= b^3 - b(a(a+1) + a(a+2) + (a+1)(a+2))$$

$$B_3 = (A_2 \times b - B_2 \times (a+2))$$

$$= b(a+a+1) \times b - (a(a+1) - b^2) \times (a+2)$$

$$= b^2(a+a+1+a+2) - a(a+1)(a+2)$$

We know that the non-repeating combination of *n* elements taken *p* by *p* is equal to :

$$C_n^p = \frac{n!}{p! \times (n-p)!}$$

Considering the function $a \to K_n^p(a)$ $(a \in \mathbb{R})$ which sums the multiplications between all the elements of the non-repeating combinations of the *n* elements $\{a; a+1; a+2; ....; a+n-1 : n \in \mathbb{N}^*\}$ taken *p* by *p* :

$K_2^1(a) = (a) + (a+1)$; $\qquad C_2^1 = \frac{2!}{1!\times(2-1)!} = 2$

$K_2^2(a) = a(a+1)$; $\qquad C_2^2 = \frac{2!}{2!\times(2-2)!} = 1$

$K_3^2(a) = a(a+1) + a(a+2) + (a+1)(a+2)$; $\qquad C_3^2 = \frac{3!}{2!\times(3-2)!} = 3$

$K_3^1(a) = (a) + (a+1) + (a+2)$; $\qquad C_3^1 = \frac{3!}{1!\times(3-1)!} = 3$

$K_3^3(a) = a(a+1)(a+2)$; $\qquad C_3^3 = \frac{3!}{3!\times(3-3)!} = 1$

We'll have :

$A_1 = -bK_1^0(a)$; $B_1 = -K_1^1(a)$

$A_2 = bK_2^1(a)$; $B_2 = -(b^2K_2^0(a) - K_2^2(a)$

$A_3 = b^3K_3^0(a) - bK_3^2(a)$; $B_3 = b^2K_3^1(a) - bK_3^3(a)$

$A_4 = -(b^3K_4^1(a) - bK_4^3(a))$; $B_4 = b^4K_4^0(a) - b^2K_4^2(a) + K_4^4(a)$

$A_5 = -(b^5K_5^0(a) - b^3K_5^2(a) + bK_5^4(a))$; $B_5 = -(b^4K_5^1(a) - b^2K_5^3(a) + K_5^5(a))$

$A_6 = b^5K_6^1(a) - b^3K_6^3(a) + bK_6^5(a)$; $B_6 = -(b^6K_6^0(a) - b^4K_6^2(a) + b^2K_6^4(a) - K_6^6(a))$

We observe that A_k and B_k can be written in terms of the parity of the derivation order k :

$\forall\, k \in \mathbb{N}^* si\ k = 2n+1 : n \in \mathbb{N}$:

$$A_{2n+1} = \sum_{p=0}^{n} (-1)^{p+n+1} b^{2(n-p)+1} K_{2n+1}^{2p}(a)$$

$$B_{2n+1} = \sum_{p=0}^{n} (-1)^{p+n+1} b^{2(n-p)} K_{2n+1}^{2p+1}(a) \qquad (4)$$

$\forall\, k \in \mathbb{N}^*\ si\ k = 2n\ : n \in \mathbb{N}^*$:

$$A_{2n} = \sum_{p=1}^{n} (-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a)$$

$$B_{2n} = \sum_{p=0}^{n} (-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a)$$

Demonstration by recurrence :

For $k = 1$:

$$A_1 = \textstyle\sum_{p=0}^{0} (-1)^{p+n+1} b^{2(n-p)+1} K_{2n+1}^{2p}(a) = -b\, K_1^0(a) = -b$$

$$B_1 = \textstyle\sum_{p=0}^{0} (-1)^{p+n+1} b^{2(n-p)} K_{2n+1}^{2p+1}(a) = -K_1^1(a) = -a$$

For $k = 2$:

$$A_2 = \sum_{p=1}^{1}(-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a) = bK_2^1(a) = b(a+a+1)$$

$$B_2 = \sum_{p=0}^{1}(-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a) = -(b^2 K_2^0(a) - K_2^2(a)) = a(a+1) - b^2$$

Assuming : $\forall\ k \in \mathbb{N}^*\ si\ k = 2n\ : n \in \mathbb{N}^*$:

$$A_{2n} = \sum_{p=1}^{n}(-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a)$$

$$B_{2n} = \sum_{p=0}^{n}(-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a)$$

According to relation (3) :

$A_{k+1} = -(A_k \times (a+k) + B_k \times b)$

$B_{k+1} = (A_k \times b - B_k \times (a+k))$

So :

$$A_{2n+1} = -(A_{2n} \times (a+2n) + B_{2n} \times b)$$

$$= -(a+2n)\sum_{p=1}^{n}(-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a) - b\sum_{p=0}^{n}(-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a)$$

$$= (a+2n)\sum_{p=1}^{n}(-1)^{p+n+1} b^{2(n-p)+1} K_{2n}^{2p-1}(a) + \sum_{p=0}^{n}(-1)^{p+n+1} b^{2(n-p)+1} K_{2n}^{2p}(a)$$

$$= \sum_{p=1}^{n}(-1)^{p+n+1} b^{2(n-p)+1}(a+2n) K_{2n}^{2p-1}(a) + \sum_{p=1}^{n}(-1)^{p+n+1} b^{2(n-p)+1} K_{2n}^{2p}(a) + (-1)^{n+1} b^{2n+1} K_{2n+1}^{0}(a)$$

$$= \sum_{p=1}^{n}(-1)^{p+n+1} b^{2(n-p)+1}\left[(a+2n) K_{2n}^{2p-1}(a) + K_{2n}^{2p}(a)\right] + (-1)^{n+1} b^{2n+1} K_{2n+1}^{0}(a)$$

$$= \sum_{p=1}^{n}(-1)^{p+n+1} b^{2(n-p)+1}\ K_{2n+1}^{2p}(a) + (-1)^{n+1} b^{2n+1} K_{2n+1}^{0}(a)$$

$$= \sum_{p=0}^{n}(-1)^{p+n+1} b^{2(n-p)+1}\ K_{2n+1}^{2p}(a)$$

Because $(a+2n)K_{2n}^{2p-1}(a) + K_{2n}^{2p}(a) = K_{2n+1}^{2p}(a)$ (Appendix 1)

$$B_{2n+1} = (A_{2n} \times b - B_{2n} \times (a+2n))$$

$$= b\sum_{p=1}^{n}(-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a) - (a+2n)\sum_{p=0}^{n}(-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a)$$

$$= \sum_{p=0}^{n-1}(-1)^{p+n+1}b^{2(n-p)}K_{2n}^{2p+1}(a) + \sum_{p=0}^{n}(-1)^{p+n+1}b^{2(n-p)}(a+2n)K_{2n}^{2p}(a)$$

$$= \sum_{p=0}^{n-1}(-1)^{p+n+1}b^{2(n-p)}\left[K_{2n}^{2p+1}(a) + (a+2n)K_{2n}^{2p}(a)\right] + (-1)^{2n+1}(a+2n)K_{2n}^{2n}(a)$$

$$= \sum_{p=0}^{n-1}(-1)^{p+n+1}b^{2(n-p)}K_{2n+1}^{2p+1}(a) + (-1)^{2n+1}(a+2n)K_{2n}^{2n}(a)$$

$$= \sum_{p=0}^{n}(-1)^{p+n+1}b^{2(n-p)}K_{2n+1}^{2p+1}(a)$$

Because :

$K_{2n}^{2p+1}(a) + (a+2n)K_{2n}^{2p}(a) = K_{2n+1}^{2p+1}(a)$

et $(a+2n)K_{2n}^{2n}(a) = K_{2n+1}^{2n+1}(a)$ (Appendix 1)

And since we're interested in derivatives of order $(2j-1)$:

$$f^{(2j-1)}(x) = \frac{A_{2j-1}\sin\big(bln(x)\big) + B_{2j-1}\cos(bln(x))}{x^{a+2j-1}}$$

Replacing *n* by $(j-1)$ in relation (4) :

$$A_{2j-1} = \sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)$$

$$B_{2j-1} = \sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)$$

(5)

$$f^{(2j-1)}(N) = \frac{A_{2j-1}\sin\big(bln(N)\big) + B_{2j-1}\cos(bln(N))}{N^{a+2j-1}}$$

$$f^{(2j-1)}(1) = \frac{A_{2j-1}\sin\big(bln(1)\big) + B_{2j-1}\cos(bln(1))}{1^{a+2j-1}} = B_{2j-1}$$

(6)

Calculation of $g^{(2j-1)}(N)$ *et* $g^{(2j-1)}(1)$:

By doing the same for the function $g(x) = \frac{\sin(bln(x))}{x^a}$:

$$\left[\frac{\sin(bln(x))}{x^a}\right]' = \frac{bx^{a-1}\cos(bln(x)) - ax^{a-1}\sin(bln(x)}{x^{2a}} = \frac{b\cos(bln(x)) - a\sin(bln(x))}{x^{a+1}}$$

$$g^{(k)} = \frac{A'_k \sin\big(bln(x)\big) + B'_k \cos(bln(x))}{x^{a+k}}$$

$A'_1 = -a = B_1 \quad B'_1 = b = -A_1$

We show by recurrence that : $\forall k \in \mathbb{N}^* \; A'_k = B_k \;\; et \; B'_k = -A_k$

For $k = 1$ it's true.

Assuming that $\forall k \in \mathbb{N}^* \; A'_k = B_k \;\; et \; B'_k = -A_k$

$$g^{(k)} = \frac{B_k \sin\big(bln(x)\big) - A_k \cos(bln(x))}{x^{a+k}}$$

$$g^{(k+1)} = \left[\frac{B_k \sin\big(bln(x)\big) - A_k \cos(bln(x))}{x^{a+k}}\right]'$$

$$= B_k \times \left[\frac{\sin(bln(x))}{x^{a+k}}\right]' - A_k \times \left[\frac{\cos(bln(x))}{x^{a+k}}\right]'$$

$$= B_k \times \left[\frac{bcos\big(bln(x)\big) - (a+k)\sin(bln(x))}{x^{a+k+1}}\right] - A_k \times \left[\frac{-bsin\big(bln(x)\big) - (a+k)\cos(bln(x))}{x^{a+k+1}}\right]$$

$$= \frac{(B_k \times b + A_k \times (a+k))\cos(bln(x)) + (A_k \times b - B_k \times (a+k))\sin(bln(x))}{x^{a+k+1}}$$

$A'_{k+1} = A_k \times b - B_k \times (a+k)$

$B'_{k+1} = B_k \times b + A_k \times (a+k)$

We know that :

$A_{k+1} = -(A_k \times (a+k) + B_k \times b)$

$B_{k+1} = (A_k \times b - B_k \times (a+k))$

We therefore have equality :

$A'_{k+1} = B_{k+1}$ and $B'_{k+1} = -A_{k+1}$

As a result : $\forall k \in \mathbb{N}^* \; A'_k = B_k \;\; et \; B'_k = -A_k$

And the derivatives of order $(2j-1)$ of g can be calculated as follows:

$$g^{(2j-1)} = \frac{A'_{2j-1} \sin\big(bln(x)\big) + B'_{2j-1} \cos(bln(x))}{x^{a+2j-1}}$$

$$= \frac{B_{2j-1} \sin\big(bln(x)\big) - A_{2j-1} \cos(bln(x))}{x^{a+2j-1}}$$

Consequently :

$$g^{(2j-1)}(N) = \frac{B_{2j-1}\sin\big(bln(N)\big) - A_{2j-1}\cos(bln(N))}{N^{a+2j-1}} \quad (7)$$

$$g^{(2j-1)}(1) = \frac{B_{2j-1}\sin\big(bln(1)\big) - A_{2j-1}\cos(bln(1))}{1^{a+2j-1}} = -A_{2j-1}$$

Calculation of $\sum_{n=1}^{N} f(n)$ and $\sum_{n=1}^{N} g(n)$:

According to Euler-Maclaurin we have :

$$\sum_{n=1}^{N} f(n) = \frac{f(1)+f(N)}{2} + \int_1^N f(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(f^{(2j-1)}(N) - f^{(2j-1)}(1))$$

$$\sum_{n=1}^{N} g(n) = \frac{g(1)+g(N)}{2} + \int_1^N g(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(g^{(2j-1)}(N) - g^{(2j-1)}(1))$$

According to relationships (5); (6) and (7) :

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(1)\right) = \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(\frac{A_{2j-1}\sin\big(bln(N)\big)+B_{2j-1}\cos(bln(N))}{N^{a+2j-1}} - B_{2j-1}\right)$$

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(g^{(2j-1)}(N) - g^{(2j-1)}(1)\right) = \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(\frac{B_{2j-1}\sin\big(bln(N)\big)-A_{2j-1}\cos(bln(N))}{N^{a+2j-1}} + A_{2j-1}\right)$$

Posing :

- $U = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{A_{2j-1}}{N^{a+2j-1}})$
- $V = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{B_{2j-1}}{N^{a+2j-1}})$
- $W_1 = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times B_{2j-1})$
- $W_2 = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times A_{2j-1})$

We'll have :

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(1)\right) = U\sin\big(bln(N)\big) + V\cos\big(bln(N)\big) - W_1$$

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(g^{(2j-1)}(N) - g^{(2j-1)}(1)\right) = V\sin\big(bln(N)\big) - U\cos\big(bln(N)\big) + W_2$$

According to relation (2) :

$$\frac{f(1)+f(N)}{2} + \int_1^N f(x)dx = \frac{1}{2} + \frac{\cos\big(bln(N)\big)}{2N^a} + \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\cos\big(bln(N)\big)+\frac{b}{1-a}\sin\big(bln(N)\big)\right]-\frac{1}{1-a}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

$$\frac{g(1)+g(N)}{2} + \int_1^N g(x) = \frac{\sin\big(bln(N)\big)}{2N^a} + \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\sin\big(bln(N)\big)-\frac{b}{1-a}\cos\big(bln(N)\big)\right]+\frac{b}{(1-a)^2}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

Posing :

- $U' = \frac{\frac{bN^{1-a}}{(1-a)^2}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{bN^{1-a}}{(1-a)^2+b^2}$
- $V' = \frac{1}{2N^a} + \frac{\frac{N^{1-a}}{1-a}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{1}{2N^a} + \frac{(1-a)N^{1-a}}{(1-a)^2+b^2}$
- $W_1' = \frac{1}{2} - \frac{\frac{1}{1-a}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2}$
- $W_2' = \frac{\frac{b}{(1-a)^2}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{b}{(1-a)^2+b^2}$

We'll have :

$$\frac{f(1)+f(N)}{2} + \int_1^N f(x)dx = U'\sin\big(bln(N)\big) + V'\cos\big(bln(N)\big) + W_1'$$

$$\frac{g(1)+g(N)}{2} + \int_1^N g(x)dx = V'\sin\big(bln(N)\big) - U'\cos\big(bln(N)\big) + W_2'$$

This gives :

$$\sum_{n=1}^{N} f(n) = U'\sin\big(bln(N)\big) + V'\cos\big(bln(N)\big) + W_1' + U\sin\big(bln(N)\big) + V\cos\big(bln(N)\big) - W_1$$

$$= (U' + U)\sin\big(bln(N)\big) + (V' + V)\cos\big(bln(N)\big) + W_1' - W_1)$$

$\sum_{n=1}^{N} g(n) = V' \sin\big(bln(N)\big) - U' \cos\big(bln(N)\big) + W_2' + V \sin\big(bln(N)\big) - U \cos\big(bln(N)\big) + W_2$

$$= (V' + V)\sin\big(bln(N)\big) - (U' + U) \cos\big(bln(N)\big) + W_2' + W_2$$

$$\sum_{n=1}^{N} g(n) = (V' + V)\sin\big(bln(N)\big) - (U' + U) \cos\big(bln(N)\big) + W_2' + W_2$$

$$\sum_{n=1}^{N} f(n) = (U' + U) \sin\big(bln(N)\big) + (V' + V)\cos\big(bln(N)\big) + W_1' - W_1 \qquad (8)$$

Note that :

$$\left[\sum_{n=1}^{N} f(n) - (W_1' - W_1)\right]^2 + \left[\sum_{n=1}^{N} g(n) - (W_2' + W_2)\right]^2 = (U' + U)^2 + (V' + V)^2$$

Since we always have :

$[A \sin(x) + B \cos(x)]^2 + [B \sin(x) - A \cos(x)]^2 = A^2 + B^2$

We know that the general equation of a spiral of variable radius r and center *(u,v)* in the Cartesian plane can be written as follows:

$[x - u]^2 + [y - v]^2 = r^2$

Replacing x by $\sum_{n=1}^{N} f(n)$ and y by $-\sum_{n=1}^{N} g(n)$ in the equation, we conclude that :

$\forall (a, b) \in \mathbb{R}^2\ avec\ a \neq 1\ , Z(s) = \sum_{n=1}^{N} f(n) - i \times \sum_{n=1}^{N} g(n)$ in the complex plane is a spiral of radius $r = \sqrt{(U' + U)^2 + (V' + V)^2}$ and center $(W_1' - W_1\ , -(W_2' + W_2))$. This corroborates our observation.

$f(n) = \frac{\cos(bln(n))}{n^a}$ and $g(n) = \frac{\sin(bln(n))}{n^a}$

$$r = \sqrt{(U' + U)^2 + (V' + V)^2}$$

$$=$$

$$\sqrt{\left(\frac{bN^{1-a}}{(1-a)^2+b^2}+\ \sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{A_{2j-1}}{N^{a+2j-1}}\right)\right)^2+\left(\frac{1}{2N^a}+\frac{(1-a)N^{1-a}}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{B_{2j-1}}{N^{a+2j-1}}\right)\right)^2}$$

$$(W_1'-W_1\,,-(W_2'+W_2))$$

$$=$$

$$\left(\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}B_{2j-1}\right),\frac{-b}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}A_{2j-1}\right)\right)$$

From (1):

$$\zeta(a+ib)=\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\cos(bln(n))}{n^a}-i\times\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\sin(bln(n))}{n^a}$$

$$=\lim_{N\to+\infty}\sum_{n=1}^{N}f(n)-i\times\lim_{N\to+\infty}\sum_{n=1}^{N}g(n)$$

$$\lim_{N\to+\infty}\sum_{n=1}^{N}f(n)\ =\lim_{N\to+\infty}(U'+\ U)\sin\big(bln(N)\big)+(V'+V)\cos\big(bln(N)\big)+\ W_1'-W_1$$

$$\lim_{N\to+\infty}\sum_{n=1}^{N}g(n)\ =\lim_{N\to+\infty}(V'+V)\sin\big(bln(N)\big)-(U'+\ U)\cos\big(bln(N)\big)+W_2'+W_2$$

We know that :

- $-1<\sin\big(bln(N)\big)<1$ and $-1<\cos\big(bln(N)\big)<1$
- $U'+\ U=\frac{bN^{1-a}}{(1-a)^2+b^2}\ +\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{A_{2j-1}}{N^{a+2j-1}}\right)$
- $V'+V=\frac{1}{2N^a}+\frac{(1-a)N^{1-a}}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{B_{2j-1}}{N^{a+2j-1}}\right)$
- $W_1=\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times B_{2j-1}\right)$
- $W_1'=\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}$
- $W_2=\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times A_{2j-1}\right)$
- $W_2'=\frac{b}{(1-a)^2+b^2}$

- $A_{2j-1} = \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)$
- $B_{2j-1} = \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)$

When $a > 1$:

$\lim_{N\to+\infty} (U' + U) = 0$

$\lim_{N\to+\infty} (V' + V) = 0$

So :

$$\lim_{N\to+\infty} \sum_{n=1}^{N} f(n) = W_1' - W_1$$

$$= \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty} \left[\frac{b_{2j}}{(2j)!} \times B_{2j-1}\right]$$

$$= \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty} \left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$$

$$\lim_{N\to+\infty} \sum_{n=1}^{N} g(n) = W_2' + W_2$$

$$= \frac{b}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty} \left[\frac{b_{2j}}{(2j)!} \times A_{2j-1}\right]$$

$$= \frac{b}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty} \left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$$

We know that the zeta function when $a > 1$ converges.

So $\sum_{n=1}^{N} f(n)$ and $\sum_{n=1}^{N} g(n)$ also converge, and the zeta function can be written as :

$\zeta(a + ib) = u(a, b) + iv(a, b)$

with $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty} \left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$

and $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty} \left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

This result can be interpreted as :

- When $N \to +\infty$ the radius of the spiral tends towards 0, producing a point on the complex plane with coordinates $(u(a,b), v(a,b))$.

When $a < 1$:

$\sum_{n=1}^{N} f(n)$ and $\sum_{n=1}^{N} g(n)$ do not admit a specific limit, but as the function *Z(s)* diverges it graphically represents on the complex plane a spiral of radius $r = \sqrt{(U' + U)^2 + (V' + V)^2}$ tending towards $+\infty$ and with a coordinate center $(u(a,b), v(a,b))$:

- $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$
- $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

We define the function $s \to Y(s), \forall s \in \mathbb{C} - \{1\}, s = a + ib$, tel que :

$$Y(s) = u(a,b) + iv(a,b) \qquad (9)$$

- $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$
- $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

Note that :

- $\zeta(s)$ et $Y(s)$ are holomorphic on $\mathbb{C} - \{1\}$
- $\zeta(s) = Y(s)$ on $\{s = a + ib \in \mathbb{C} - \{1\} : a > 1\}$
- $\{s = a + ib \in \mathbb{C} - \{1\} : a > 1\}$ is a part of $\mathbb{C} - \{1\}$

So according to the principle of analytical extension $\zeta(s) = Y(s)$ on $\mathbb{C} - \{1\}$.

This means that assumption (0) is true and we can write :

$$\forall s \in \mathbb{C} - \{1\}, s = a + ib : \zeta(s) = u(a,b) + iv(a,b)$$

- $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$
- $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

Calculation of $\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$:

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$$

$$=$$

$$-\frac{b_2}{2!}K_1^1(a) + \frac{b_4}{4!}b^2K_3^1(a) - \frac{b_4}{4!}K_3^3(a) - \frac{b_6}{6!}b^4K_5^1(a) + \frac{b_6}{6!}b^2K_5^3(a) - \frac{b_6}{6!}K_5^5(a) + \frac{b_8}{8!}b^6K_7^1(a) - \frac{b_8}{8!}b^4K_7^3(a) + \frac{b_8}{8!}b^2K_7^5(a) - \frac{b_8}{8!}K_7^7(a) \dots$$

Calculation of $\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$:

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$$

$$=$$

$$-\frac{b_2}{2!}bK_1^0(a) - \frac{b_4}{4!}bK_3^2(a) + \frac{b_4}{4!}b^3K_3^0(a) - \frac{b_6}{6!}bK_5^4(a) + \frac{b_6}{6!}b^3K_5^2(a) - \frac{b_6}{6!}b^5K_5^0(a) - \frac{b_8}{8!}bK_7^6(a) + \frac{b_8}{8!}b^3K_7^4(a) - \frac{b_8}{8!}b^5K_7^2(a) + \frac{b_8}{8!}b^7K_7^0(a) \dots$$

We rearrange the terms of the two expressions:

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$$

$$=$$

$$-\left[\frac{b_2}{2!}K_1^1(a) + \frac{b_4}{4!}K_3^3(a) + \frac{b_6}{6!}K_5^5(a) + \frac{b_8}{8!}K_7^7(a) \dots\right]$$

$$+b^2\left[\frac{b_4}{4!}K_3^1(a) + \frac{b_6}{6!}K_5^3(a) + \frac{b_8}{8!}K_7^5(a) \dots\right]$$

$$-b^4\left[\frac{b_6}{6!}K_5^1(a)+\frac{b_8}{8!}K_7^3(a)\ldots\right]$$

$$+b^6\left[\frac{b_8}{8!}K_7^1(a)\ldots\right]$$

$$\vdots$$

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)\right]$$

$$=$$

$$-b\left[\frac{b_2}{2!}K_1^0(a)+\frac{b_4}{4!}K_3^2(a)+\frac{b_6}{6!}K_5^4(a)+\frac{b_8}{8!}K_7^6(a)\ldots\right]$$

$$+b^3\left[\frac{b_4}{4!}K_3^0(a)+\frac{b_6}{6!}K_5^2(a)+\frac{b_8}{8!}K_7^4(a)\ldots\right]$$

$$-b^5\left[\frac{b_6}{6!}K_5^0(a)+\frac{b_8}{8!}K_7^2(a)\ldots\right]$$

$$+b^7\left[\frac{b_8}{8!}K_7^0(a)\ldots\right]$$

$$\vdots$$

We observe that :

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)\right]$$

$$=$$

$$-\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]$$

And :

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)\right]$$

$$=$$

$$-\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]$$

Summary :

$\forall s \in \mathbb{C} - \{1\}, s = a + ib$: (10)

$$\boldsymbol{\zeta(s)} = \left[\tfrac{1}{2} - \tfrac{1-a}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty}\left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \tfrac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i} \times \left[\tfrac{-b}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1} \sum_{j=n+1}^{+\infty} \tfrac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

5 Consequences

5.1 $b = 0$

According to relationship (10), when $b = 0$:

$$\zeta(a) = \frac{1}{2} - \frac{1}{1-a} + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2j-1}(a) = \frac{1}{2} - \frac{1}{1-a} + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} \prod_{p=0}^{2(j-1)} (a+p)$$

We can check that :

- $\zeta(0) = -\frac{1}{2}$
- $\zeta(-1) = \frac{1}{2} - \frac{1}{2} - \frac{b_2}{2} = -\frac{1}{12} \quad (b_2 = \frac{1}{6})$
- $\zeta(-2) = \frac{1}{2} - \frac{1}{3} - b_2 = 0$
- Generally, we find that $\zeta(-2n) = 0$ (trivial zeros)

We know that $\forall n \in \mathbb{N}$:

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1} b_{2n}}{(2n)!} \pi^{2n} \quad \text{and} \quad \zeta(-n) = (-1)^n \frac{b_{n+1}}{n+1}$$

So we have the following two equalities $\forall n \in \mathbb{N}$:

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} \prod_{p=0}^{2(j-1)} (2n+p) = (-1)^{n+1} \frac{2^{2n-1} b_{2n}}{(2n)!} \pi^{2n} - \frac{1}{2n-1} - \frac{1}{2}$$

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} \prod_{p=0}^{2(j-1)} (p-n) = (-1)^{n} \frac{b_{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{2}$$

5.2 $b \neq 0$

The equation :

$$\zeta(s) = 0$$

According to relation (10), Implies :

- $\frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a) \right] = 0$
- $\frac{-b}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n+1} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a) \right] = 0$

Implies :

- $\sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a) \right] = \frac{1-a}{(1-a)^2+b^2} - \frac{1}{2}$
- $\sum_{n=0}^{+\infty} \left[(-1)^n b^{2n+1} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a) \right] = \frac{b}{(1-a)^2+b^2}$

Implies, by dividing by b in the second equation and replacing $\frac{1}{(1-a)^2+b^2}$ in the first equation:

$$\sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a) \right]$$

$$=$$

$$(1-a) \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a) \right] - \frac{1}{2}$$

Implies :

$$\frac{1}{2} + \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} \left[K_{2j-1}^{2(j-n)-1}(a) - (1-a) K_{2j-1}^{2(j-n-1)}(a) \right] \right] = 0$$

So we can say that $s = a + ib$ is a non-trivial zero of the zeta function when a and b are solutions of the equation :

$$\frac{1}{2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2(j-n)-1}(a)-(1-a)K_{2j-1}^{2(j-n-1)}(a)\right]\right]=0$$

5.3 Proof of the Riemann hypothesis

The Riemann hypothesis can be stated as follows:

$$\zeta(s)=0\ \ Et\ \ 0<a<1\,, implique\ a=\frac{1}{2}$$

According to relation (10) :

$$\boldsymbol{\zeta(s)}=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i}\times\left[\frac{-b}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

$\boldsymbol{\zeta(s)=0}$

Implies :

- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]+\frac{1}{2}=\frac{1-a}{(1-a)^2+b^2}$
- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]=\frac{b}{(1-a)^2+b^2}$

Note that :

$$\frac{\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]}{\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]+\frac{1}{2}}=\frac{b}{1-a}$$

Posing $u=\frac{b}{1-a}=\tan(\frac{\theta}{2})$

So :

$$\frac{\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]}{1+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]-\frac{1}{2}}=\tan\left(\frac{\theta}{2}\right)=\frac{\sin(\theta)}{1+\cos(\theta)}$$

Implies :

- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]=\sin(\theta)=\frac{2u}{1+u^2}$
- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]-\frac{1}{2}=\cos(\theta)=\frac{1-u^2}{1+u^2}$

And since :

- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]+\frac{1}{2}=\frac{1-a}{(1-a)^2+b^2}$
- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]=\frac{b}{(1-a)^2+b^2}$

So :

- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]-\frac{1}{2}=\frac{1-a}{(1-a)^2+b^2}-1=\frac{\frac{1}{1-a}-1-\left(\frac{b}{1-a}\right)^2}{1+\left(\frac{b}{1-a}\right)^2}$
- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]=\frac{b}{(1-a)^2+b^2}=\frac{\frac{b}{(1-a)^2}}{1+\left(\frac{b}{1-a}\right)^2}$

Implies :

- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]-\frac{1}{2}=\frac{\frac{1}{1-a}-1-u^2}{1+u^2}$
- $\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]=\frac{\frac{1}{1-a}u}{1+u^2}$

Implies :

- $\frac{\frac{1}{1-a}-1-u^2}{1+u^2}=\frac{1-u^2}{1+u^2}$
- $\frac{\frac{1}{1-a}u}{1+u^2}=\frac{2u}{1+u^2}$

Implies :

$\frac{1}{1-a}=2$

Implies :

$a=\frac{1}{2}$

5.4 Assumption (1)

Here I'm going to use hypothesis (1) mentioned in Appendix 1, linked to one of the properties of the function $a \to K_n^p(a) : a \in \mathbb{R}\,,\ (n,p) \in \mathbb{N}^2 et\ n \geq p$ function, which I find to be true but do not hold a proof (Appendix 1):

$$\forall p \in \mathbb{N} :\ \left[K_p^p(a)\right]^{(n)} = n!\,K_p^{p-n}(a)$$

(n) is the derivative of order *n*, $n \in \mathbb{N}$

So :

$$\left[K_{2j-1}^{2j-1}(a)\right]^{(2n)} = (2n)!\,K_{2j-1}^{2j-1-2n}(a) = (2n)!\,K_{2j-1}^{2(j-n)-1}(a)$$

$$\left[K_{2j-1}^{2j-1}(a)\right]^{(2n+1)} = (2n+1)!\,K_{2j-1}^{2j-1-2n-1}(a) = (2n+1)!\,K_{2j-1}^{2(j-n-1)}(a)$$

Posing $\forall a \in \mathbb{R}\,,\ L(a) = -\frac{1}{2} + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2j-1}(a)$

Note that :

$$\frac{[L(a)]^{(2n)}}{(2n)!} = \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a)$$

$$\frac{[L(a)]^{(2n+1)}}{(2n+1)!} = \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a)$$

By replacing $\sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a)$ and $\sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a)$ in relation (10), we get :

$$\boldsymbol{\zeta(s)} = 1 - \frac{1-a}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty} (-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}(a) + i\left[\sum_{n=0}^{+\infty} (-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}(a) - \frac{b}{(1-a)^2+b^2}\right]$$

$$\boldsymbol{\zeta(s) = 0}$$

Implies :

- $1-\frac{1-a}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n}}{(2n)!}L^{(2n)}(a)=0$

- $\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)!}L^{(2n+1)}(a)-\frac{b}{(1-a)^2+b^2}=0$

Implies :

- $\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n}}{(2n)!}L^{(2n)}(a)=\frac{1-a}{(1-a)^2+b^2}-1=\frac{\frac{1}{1-a}-1-\left(\frac{b}{1-a}\right)^2}{1+\left(\frac{b}{1-a}\right)^2}$

- $\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)!}L^{(2n+1)}(a)=\frac{b}{(1-a)^2+b^2}=\frac{\frac{b}{(1-a)^2}}{1+\left(\frac{b}{1-a}\right)^2}$

Posing $u=\frac{b}{1-a}=\tan(\frac{\theta}{2})$:

- $\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n}}{(2n)!}L^{(2n)}(a)=\frac{\frac{1}{1-a}-1-u^2}{1+u^2}$

- $\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)!}L^{(2n+1)}(a)=\frac{\frac{1}{1-a}u}{1+u^2}$

- $\frac{\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)!}L^{(2n+1)}(a)}{1+\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n}}{(2n)!}L^{(2n)}(a)}=\tan\left(\frac{\theta}{2}\right)=\frac{\sin(\theta)}{1+\cos(\theta)}$

Then :

$$\left[\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n}}{(2n)!}L^{(2n)}(a)\right]^2+\left[\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)!}L^{(2n+1)}(a)\right]^2=1$$

So :

$$\left[\frac{\frac{1}{1-a}-1-u^2}{1+u^2}\right]^2+\left[\frac{\frac{1}{1-a}u}{1+u^2}\right]^2=1$$

Implies :

$$\left(\frac{1}{1-a}-1\right)^2-2\left(\frac{1}{1-a}-1\right)u^2+u^4+\left(\frac{1}{1-a}u\right)^2=1+2u^2+u^4$$

Implies :

$$\left(\frac{1}{1-a}-1\right)^2-1+u^2\left(\left(\frac{1}{1-a}\right)^2-\frac{2}{1-a}\right)=0$$

Implies :

- $\left(\frac{1}{1-a}-1\right)^2-1=0$
- $\left(\frac{1}{1-a}\right)^2-\frac{2}{1-a}=0$

Because $\left(\frac{1}{1-a}-1\right)^2-1$ and $\left(\frac{1}{1-a}\right)^2-\frac{2}{1-a}$ have the same sign on either side of $\frac{1}{2}$

And $u^2=\left(\frac{b}{1-a}\right)^2>0$, $b\neq 0$

Implies :

$$\left(\frac{1}{1-a}\right)^2=\frac{2}{1-a}$$

Implies :

$$\frac{1}{1-a}=2$$

Implies :

$$a=\frac{1}{2}$$

Conclusion

Based on the results of this article, we can define the zeta function on all the complex plane except in $s = 1$ by :

$$\forall s = a + ib \in \mathbb{C} - \{1\}$$

$$\boldsymbol{\zeta(s)} = \left[\frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty}\left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i} \times \left[\frac{-b}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

We then prove that Riemann's hypothesis is true and that all non-trivial zeros of the zeta function have a real part $a = \frac{1}{2}$ and an imaginary part *b* that satisfies the equation :

$$\frac{1}{2} + \sum_{n=0}^{+\infty}\left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2(j-n)-1}\left(\frac{1}{2}\right) - \frac{1}{2}K_{2j-1}^{2(j-n-1)}\left(\frac{1}{2}\right)\right]\right] = 0$$

Assuming that assumption (1) in Appendix 1 is true, we can write :

$$\boldsymbol{\zeta(s)} = 1 - \frac{1-a}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty}(-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}(a) + i\left[\sum_{n=0}^{+\infty}(-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}(a) - \frac{b}{(1-a)^2+b^2}\right]$$

With : $L(a) = -\frac{1}{2} + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2j-1}(a)$

This means that when $a = \frac{1}{2}$ the imaginary part *b* of all non-trivial zeros of the zeta function is a solution of equation :

$$\left[\sum_{n=0}^{+\infty}(-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}\left(\frac{1}{2}\right)\right]^2 + \left[\sum_{n=0}^{+\infty}(-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}\left(\frac{1}{2}\right)\right]^2$$

Appendix 1

In this article we have defined the function noted $a \rightarrow K_n^p(a)$, $(a \in \mathbb{R})$, $(n, p) \in \mathbb{N}^2 et\ n \geq p$which sums the multiplications between all the elements of the non-repeating combinations of the n elements $\{a; a+1; a+2; \dots; a+n-1 : n \in \mathbb{N}\}$ taken p by p :

$$C_n^p = \frac{n!}{p! \times (n-p)!}$$

The properties of this function include :

- $K_{2n}^{2p+1}(a) + (a+2n)K_{2n}^{2p}(a) = K_{2n+1}^{2p+1}(a)$
- $K_{2n}^{2p}(a) + (a+2n)K_{2n}^{2p-1}(a) = K_{2n+1}^{2p}(a)$
- $(a+2n)K_{2n}^{2n}(a) = K_{2n+1}^{2n+1}(a)$

We must therefore prove that $\forall (n, p) \in \mathbb{N}^2 et\ n \geq p$:

$K_n^{p+1}(a) + (a+n)K_n^p(a) = K_{n+1}^{p+1}(a)$ and $(a+n)K_n^n(a) = K_{n+1}^{n+1}(a)$

We know that :

$$C_n^{p+1} + C_n^p = \frac{n!}{(p+1)! \times (n-p-1)!} + \frac{n!}{p! \times (n-p)!}$$

$$= \frac{(n!)(n-p) + (n!)(p+1)}{(p+1)! \times (n-p)!}$$

$$= \frac{(n!)(n-p) + n!(p+1)}{(p+1)! \times (n-p)!}$$

$$= \frac{(n!)(n+1)}{(p+1)! \times (n-p)!}$$

$$= \frac{(n+1!)}{(p+1)! \times (n-p)!} = C_{n+1}^{p+1}(a)$$

And $C_n^n = C_{n+1}^{n+1}$

And since $(a+n)$ it is the n+1 element of the set :

$\{a; a+1; a+2; \dots; a+n : n \in \mathbb{N}\}$

We therefore have the equivalence :

$K_n^{p+1}(a) + (a+n)K_n^p(a) = K_{n+1}^{p+1}(a)$ And $(a+n)K_n^n(a) = K_{n+1}^{n+1}(a)$

It has also been assumed that assumption (1), which can be stated as follows, is true:

$$\forall p \in \mathbb{N} : \left[K_p^p(a)\right]^{(n)} = n!\, K_p^{p-n}(a)$$

(n) is the derivative of order n, $n \in \mathbb{N}$

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