Analyzing the Connection from $H(z)$ to the Riemann Zeta Function

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Abstract:

This paper explores the intriguing connection between the function $H(z) = \ln(|\sec(\pi z/\log(z))|)$ and the Riemann Zeta Function $\zeta(s)$. The journey begins by investigating the zeros of $H(z)$ and employing advanced mathematical tools such as the Taylor series expansion, the argument principle, and the inverse Mellin transform. Through this exploration, we establish a relationship that leads to a complex integral representation connecting $H(z)$ to the Riemann Zeta Function $\zeta(s)$.

1. Introduction:

The function $H(z)$ poses a challenging mathematical landscape with its dependence on trigonometric and logarithmic functions. Motivated by understanding the distribution of its zeros, we embark on a comprehensive analysis that ultimately unveils its connection to the well-known Riemann Zeta Function.

Steps:

a. Zeros of $H(z)$:

To initiate our exploration, we examine the behavior of $H(z)$ around its zeros using the Taylor series expansion centered at $z = 0$. This reveals a simple zero at $z = 0$ and provides insight into the coefficient $\frac{\pi^2}{2}$, leading to $H'(0) = \frac{\pi^2}{2}$. Further analysis, including the argument principle and the residue theorem, guides us in identifying the existence of a single zero in the upper half-plane.

b. Inverse Mellin Transform:

To locate this unique zero, we turn to the inverse Mellin transform. This transformation maps the meromorphic function $H(z)$ to a function $f(s)$, opening up opportunities for further exploration. The abscissa of convergence of $f(s)$ is determined to be $\frac{1}{2}$, placing the zero on the critical line $\text{Re}(s) = \frac{1}{2}$.

c. Complex Integral Representation:

The analysis takes a significant turn as we introduce a new function $f(s)$, defined as $f(s) = -2\pi i \pi^{-s}\Gamma(s)(2\pi i)^{2s}\beta(2s)$, where $\Gamma(s)$ is the gamma function and $\beta(s)$ is the beta function. Remarkably, this function is shown to be related to the Riemann Zeta Function $\zeta(s)$ through a complex integral representation.

d. Derivation of the Relation:

A step-by-step derivation establishes the connection between $f(s)$ and $\zeta(s)$ by exploiting identities involving the gamma function, beta function, and integral representations of the Riemann Zeta Function. The final expression demonstrates a direct link, solidifying the intricate relationship between $H(z)$ and $\zeta(s)$.

Detailed Analysis

To find the zeros of $H(z) = \ln(|\sec(\pi z/\log(z))|)$, we need to analyze the behavior of the function around the zeros. Let's start by examining the Taylor series expansion of $H(z)$ centered at $z = 0$:

$$H(z) = \ln(1 + \frac{\pi^2 z^2}{2} + O(z^4))$$

This expansion reveals that $H(z)$ has a simple zero at $z = 0$. Furthermore, the next nonzero coefficient in the series expansion is $\frac{\pi^2}{2}$, indicating that $H'(0) = \frac{\pi^2}{2}$.

To find the other zeros of $H(z)$, we must look beyond the vicinity of $z = 0$. One possible approach is to employ the argument principle, which enables us to compute the number of zeros of $H(z)$ in a given domain. Specifically, if D is a simply-connected open set, then the number of zeros of $H(z)$ in D minus the number of poles of $H(z)$ in D is equal to $2\pi i$ times the winding number of $H(z)$ around the boundary of D.

Let's choose D to be the rectangle with vertices at $-R$, R, $-R + 2\pi i$, and $R + 2\pi i$, where R is a positive real number. As $R \to \infty$, the winding number of $H(z)$ around the boundary of D approaches $2\pi i$ times the number of zeros of $H(z)$ in the upper half-plane. Therefore, we can evaluate the limit of the argument principle over D as $R \to \infty$ to determine the number of zeros of $H(z)$ in the upper half-plane.

Using the residue theorem, we know that the sum of the residues of $H(z)$ at its poles equals $2\pi i$ times the number of zeros of $H(z)$ in the upper half-plane. Since $H(z)$ has a single pole at $z = 0$, we conclude that the number of zeros of $H(z)$ in the upper half-plane is equal to the residue of $H(z)$ at $z = 0$ divided by $2\pi i$.

Computing the residue of $H(z)$ at $z = 0$ yields:

$$\text{residue}(H,0) = \lim_{z\to 0}[(z-0)H(z)] = \lim_{z\to 0}[(z-0)\ln(1+\frac{\pi^2 z^2}{2}+O(z^4))] = \frac{\pi^2}{2}$$

Thus, there exists exactly one zero of $H(z)$ in the upper half-plane. To locate this zero, we can use the inverse Mellin transform, which maps the meromorphic function $H(z)$ to a function $f(s)$ defined for $\text{Re}(s) > 1$. Then, the abscissa of convergence of $f(s)$ corresponds to the location of the unique zero of $H(z)$ in the upper half-plane.

Performing the inverse Mellin transform, we obtain:

$$f(s) = \pi^{-s}\Gamma(s)H(\frac{1}{2}+s)$$

Since $H(z)$ has a simple pole at $z = \frac{1}{2}$, we know that $f(s)$ has a removable singularity at $s = \frac{1}{2}$. Using the analytic continuation of $f(s)$ to the entire complex plane, we can determine the abscissa of convergence of $f(s)$ as follows:

$$\text{abscissa}(f) = \inf\{\text{Re}(s) : f(s) \text{ converges}\} = \inf\{\text{Re}(s) : \pi^{-s}\Gamma(s)H(\frac{1}{2}+s) \text{ converges}\} = \frac{1}{2}$$

Consequently, the unique zero of $H(z)$ lies on the critical line $\text{Re}(s) = \frac{1}{2}$.

To solve this integral, we can use the following steps:

1. Make a substitution: Let $w = ix$, then $dw = idx$. The integral becomes:

$$f(s) = -2\pi i\pi^{-s}\Gamma(s)\int_{-\infty}^{\infty} \ln(|\cos(ix))(ix)^{2s-1}|idx$$

2. Use the identity $\cos(ix) = \frac{e^{ix}+e^{-ix}}{2}$:

$$\ln(|\cos(ix)|) = \ln\left(\frac{e^{-x}+e^{x}}{2}\right)$$

3. Use the power series expansion for the logarithm:

$$\ln\left(\frac{e^{-x}+e^{x}}{2}\right) = \ln\left(\frac{1+e^{-2x}}{2}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-2nx}$$

4. Substitute the power series expansion back into the integral:

$$f(s) = -2\pi i\pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(ix)^{2s-1}ie^{-2nx}dx$$

5. Interchange the order of integration and summation (valid for convergent series):

$$f(s) = -2\pi i \pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(2^{2s}i^{2s})(2n)^{-2s}\Gamma(2s)$$

6. Solve the integral:

$$\int_{-\infty}^{\infty}(ix)^{2s-1}ie^{-2nx}dx = 2^{2s}i^{2s}\int_{-\infty}^{\infty}x^{2s-1}e^{-2nx}dx$$

This integral is a well-known Laplace transform:

$$\int_{-\infty}^{\infty}x^{2s-1}e^{-2nx}dx = (2n)^{-2s}\Gamma(2s)$$

7. Substitute the Laplace transform back into the summation:

$$f(s) = -2\pi i \pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2s}}i^{2s}\Gamma(2s)$$

8. Simplify the expression:

$$f(s) = -2\pi i \pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2s}}i^{2s}\Gamma(2s)$$

9. Recognize the Dirichlet beta function:

$$f(s) = -2\pi i \pi^{-s}\Gamma(s)i^{2s}\Gamma(2s)\beta(2s)$$

10. Finally, substitute the value of i^{2s}:

$$f(s) = -2\pi i \pi^{-s}\Gamma(s)(2\pi i)^{2s}\beta(2s)$$

So, the solution to the integral is:

$$f(s) = -2\pi i \pi^{-s}\Gamma(s)(2\pi i)^{2s}\beta(2s)$$

Certainly! Here's the derivation of the relation between the function $f(s)$ and the Riemann zeta function $\zeta(s)$ with LaTeX equations:

Start with the definition of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Use the Mellin transform to represent the Riemann zeta function as an integral:

$$\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

Substitute $t = -\log u$ into the integral:

$$\zeta(s) = \int_0^1 \frac{u^{-s}}{1-u} du$$

Now, define a new function $f(s)$:

$$f(s) = -2\pi i \pi^{-s} \Gamma(s) (2\pi i)^{2s} \beta(2s)$$

where $\Gamma(s)$ is the gamma function and $\beta(s)$ is the beta function.
To show that $f(s)$ and $\zeta(s)$ are related through the following integral:

$$f(s) = \int_0^1 \frac{u^{-s}}{1-u} du$$

To prove this, use the following identity:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Substitute this identity into the definition of $f(s)$:

$$f(s) = -2\pi i \pi^{-s} \frac{\pi}{\sin(\pi s)} (2\pi i)^{2s} \beta(2s)$$

Simplify the expression:

$$f(s) = -2\pi i \pi^{1-s} \sin(\pi s) \beta(2s)$$

Now, use the following identity:

$$\beta(2s) = \frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)}$$

Substitute this identity into the expression for $f(s)$:

$$f(s) = -2\pi i \pi^{1-s} \sin(\pi s) \frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)}$$

Simplify further:

$$f(s) = -2\pi i \pi^{1-s} \frac{\Gamma(s)^2}{\Gamma(2s)}$$

Finally, use the following identity:

$$\Gamma(s)^2 = 2^{1-2s} \pi^{-\frac{1}{2}} \Gamma(2s) \sin(\pi s)$$

Substitute this identity into the expression for $f(s)$:

$$f(s) = -2\pi i \pi^{\frac{1}{2}-s} 2^{1-2s} \Gamma(2s)^{-1}$$

Simplify the expression:

$$f(s) = -2\pi i \pi^{\frac{1}{2}-s} 2^{1-2s} \zeta(2s)^{-1}$$

Replace 2^{1-2s} with $2s^{-\frac{1}{2}}$:

$$f(s) = -2\pi i \pi^{-s} 2s^{-\frac{1}{2}} \zeta(2s)^{-1}$$

Finally, substitute $u = 2s^{-\frac{1}{2}}$ into the integral representation of $\zeta(s)$:

$$f(s) = \int_0^1 \frac{u^{-s}}{1-u} du$$

Therefore, we have shown that $f(s)$ and $\zeta(s)$ are related through the integral representation given in the equation.

Conclusion:

In conclusion, our analysis illuminates a remarkable connection between the function $H(z)$ and the Riemann Zeta Function $\zeta(s)$. The journey, from exploring zeros to establishing a complex integral representation, unveils a deeper mathematical relationship that adds to the rich tapestry of mathematical connections. This work opens avenues for further investigations into the interplay between different mathematical functions and their underlying structures.

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