Continued Fraction Generalization Vol. 3

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Abstract

This is a list of ten types of continued fraction generalization.
(This is vol. 3 , every volume contains 10 formulas)

I am using Euler's continued fraction formula
in order to find some nice continued fraction generalization.

The Formulas

Listed below are ten types of continued fraction generalization.
Some of which even have three variables.
On some variables you can even use complex numbers.

Formula No. 21

$$n+1-x+\cfrac{(n+1)x}{n+2-x+\cfrac{(n+2)x}{n+3-x+\cfrac{(n+3)x}{n+4-x+\cfrac{(n+4)x}{\ddots}}}} = \frac{(-x)^{n+1}}{\frac{n!}{e^x}-n!\sum_{k=0}^{n}\frac{(-x)^k}{k!}}-x$$

Formula No. 22

$$n+1+x-\cfrac{(n+1)x}{n+2+x-\cfrac{(n+2)x}{n+3+x-\cfrac{(n+3)x}{n+4+x-\cfrac{(n+4)x}{\ddots}}}} = \frac{x^{n+1}}{e^x n! - n!\sum_{k=0}^{n}\frac{x^k}{k!}}+x$$

Formula No. 23

$$nx+1x+1-\cfrac{(n+1)x}{nx+2x+1-\cfrac{(n+2)x}{nx+3x+1-\cfrac{(n+3)x}{nx+4x+1-\cfrac{(n+4)x}{\ddots}}}} = \frac{1}{\sqrt[x]{e}\cdot x^n n! - \sum_{k=0}^{n}\frac{x^n n!}{x^k k!}}+1$$

Formula No. 24

$$(2n+1)\cdot(2n+2)+x^2-\cfrac{(2n+1)\cdot(2n+2)\cdot x^2}{(2n+3)\cdot(2n+4)+x^2-\cfrac{(2n+3)\cdot(2n+4)\cdot x^2}{(2n+5)\cdot(2n+6)+x^2-\cfrac{(2n+5)\cdot(2n+6)\cdot x^2}{\ddots}}} = \frac{x^2}{\frac{(2n)!}{x^{2n}}\left(\frac{e^x}{2}+\frac{1}{2e^x}-\sum_{k=0}^{n}\frac{x^{2k}}{(2k)!}\right)}+x^2$$

Formula No. 25

$$(2n+0)\cdot(2n+1)+x^2-\cfrac{(2n+0)\cdot(2n+1)\cdot x^2}{(2n+2)\cdot(2n+3)+x^2-\cfrac{(2n+2)\cdot(2n+3)\cdot x^2}{(2n+4)\cdot(2n+5)+x^2-\cfrac{(2n+4)\cdot(2n+5)\cdot x^2}{\ddots}}} = \frac{x^2}{\frac{(2n-1)!}{x^{2n-1}}\left(\frac{e^x}{2}-\frac{1}{2e^x}-\sum_{k=0}^{n}\frac{x^{2k-1}}{(2k-1)!}\right)}+x^2$$

Formula No. 26

$$(n+1)\cdot(n+2)+x^2-\cfrac{(n+1)\cdot(n+2)\cdot x^2}{(n+3)\cdot(n+4)+x^2-\cfrac{(n+3)\cdot(n+4)\cdot x^2}{(n+5)\cdot(n+6)+x^2-\cfrac{(n+5)\cdot(n+6)\cdot x^2}{\ddots}}}=\frac{x^2}{\frac{n!}{x^n}\left(\frac{e^x}{2}+\frac{1}{2e^x}-\sum_{k=0}^{n}\frac{x^k}{(k)!}\right)}+x^2$$

Formula No. 27

(This is a finite continued fraction with n steps)

$$1x+(n-0)y-\cfrac{1x\cdot(n-1)y}{2x+(n-1)y-\cfrac{2x\cdot(n-2)y}{3x+(n-2)y-\cfrac{3x\cdot(n-3)y}{\ddots\,\cfrac{\ddots}{(n-1)x+2y-\cfrac{(n-1)x\cdot 1y}{(n-0)x+1y}}}}}=\frac{ny}{1-\frac{1}{(1-y/x)^n}}$$

Formula No. 28

(This is a finite continued fraction with $n-k$ steps)

$$k+2-\cfrac{1(k+2)}{k+4-\cfrac{2(k+3)}{k+6-\cfrac{\ddots}{\ddots\,\cfrac{(n-k-2)(k+(n-k-1))}{k+2(n-k-1)-\cfrac{(n-k-1)(k+(n-k))}{k+2(n-k)}}}}}=\frac{k+1}{1-\frac{1}{\binom{n+1}{k+1}}}$$

Formula No. 29

$$1^2+1\cdot 2\cdot x-\cfrac{1^2\cdot 3\cdot 4\cdot x}{2^2+3\cdot 4\cdot x-\cfrac{2^2\cdot 5\cdot 6\cdot x}{3^2+5\cdot 6\cdot x-\cfrac{3^2\cdot 7\cdot 8\cdot x}{\ddots}}}=\frac{2x}{1-\sqrt{1-4x}}$$

$x<1/4$

Formula No. 30

$$1ny-(0n-1)\cdot x+\cfrac{1ny\cdot(1n-1)\cdot x}{2ny-(1n-1)\cdot x+\cfrac{2ny\cdot(2n-1)\cdot x}{3ny-(2n-1)\cdot x+\cfrac{3ny\cdot(3n-1)\cdot x}{4ny-(3n-1)\cdot x+\cfrac{4ny\cdot(4n-1)\cdot x}{\ddots}}}}=\frac{x\sqrt[n]{y}}{\sqrt[n]{x+y}-\sqrt[n]{y}}+x$$

Euler's continued fraction formula

we will use ECFF everytime but we need to modified it a bit first

$$a_0 + a_0a_1 + a_0a_1a_2 + \ldots + a_0a_1a_2\cdots a_n = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{\ddots}{\ddots \cfrac{a_{n-1}}{1 + a_{n-1} - \cfrac{a_n}{1 + a_n}}}}}}$$

Lets replace $a_k \leftarrow \frac{x_k}{y_k}$ in ECFF and we will get this:

$$\frac{x_0}{y_0} + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1} + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1}\cdot\frac{x_2}{y_2} + \ldots + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1}\cdot\frac{x_2}{y_2}\cdots\frac{x_n}{y_n} = \cfrac{x_0 / y_0}{1 - \cfrac{x_1 / y_1}{1 + x_1 / y_1 - \cfrac{x_2 / y_2}{1 + x_2 / y_2 - \cfrac{\ddots}{\ddots \cfrac{x_{n-1} / y_{n-1}}{1 + x_{n-1} / y_{n-1} - \cfrac{x_n / y_n}{1 + x_n / y_n}}}}}}$$

Euler's modified continued fraction formula #1 (EMCFF#1)

$$\frac{x_0}{y_0} + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1} + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1}\cdot\frac{x_2}{y_2} + \ldots + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1}\cdot\frac{x_2}{y_2}\cdots\frac{x_n}{y_n} = \cfrac{x_0}{y_0 - \cfrac{y_0x_1}{y_1 + x_1 - \cfrac{y_1x_2}{y_2 + x_2 - \cfrac{\ddots}{\ddots \cfrac{y_{n-2}x_{n-1}}{y_{n-1} + x_{n-1} - \cfrac{y_{n-1}x_n}{y_n + x_n}}}}}}$$

Lets replace $x_k \leftarrow -x_k$ (first term not included) and we will get this:

Euler's modified continued fraction formula #2 (EMCFF#2)

$$\frac{x_0}{y_0} - \frac{x_0}{y_0}\cdot\frac{x_1}{y_1} + \frac{x_0}{y_0}\cdot\frac{x_1}{y_1}\cdot\frac{x_2}{y_2} - \ldots + (-1)^n\cdot\frac{x_0}{y_0}\cdot\frac{x_1}{y_1}\cdot\frac{x_2}{y_2}\cdots\frac{x_n}{y_n} = \cfrac{x_0}{y_0 + \cfrac{y_0x_1}{y_1 - x_1 + \cfrac{y_1x_2}{y_2 - x_2 + \cfrac{\ddots}{\ddots \cfrac{y_{n-2}x_{n-1}}{y_{n-1} - x_{n-1} + \cfrac{y_{n-1}x_n}{y_n - x_n}}}}}}$$

Introduction

For this vol. I am going to use mainly Euler's continued fraction formula.
Just so i'm clear, everytime you will see “EMCFF#1” or “EMCFF#2”
I am referring to the two Euler's modified continued fraction formulas writen above.
I also added (on some cases) examples for the formula at the end of the proof .

I hope you will like what I did here.
I value any feedback you can give me.

Formula No. 21

$$\frac{1}{e^x}=\frac{x^0}{0!}-\frac{x^1}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^n}{n!}+(-1)^{n+1}\frac{x^{n+1}}{(n+1)!}+\ldots$$

$$\frac{n!}{e^x}=\frac{n!x^0}{0!}-\frac{n!x^1}{1!}+\frac{n!x^2}{2!}-\frac{n!x^3}{3!}+\ldots+(-1)^n\frac{n!x^n}{n!}-(-1)^n\frac{n!x^{n+1}}{(n+1)!}+\ldots$$

$$\frac{n!}{e^x}=\left[\frac{n!x^0}{0!}-\frac{n!x^1}{1!}+\frac{n!x^2}{2!}-\frac{n!x^3}{3!}+\ldots+(-1)^n\frac{n!x^n}{n!}\right]+\left[-(-1)^n\frac{n!x^{n+1}}{(n+1)!}+(-1)^n\frac{n!x^{n+2}}{(n+2)!}-(-1)^n\frac{n!x^{n+3}}{(n+3)!}+\ldots\right]$$

$$\frac{n!}{e^x}-n!\sum_{k=0}^{n}\frac{(-x)^k}{k!}=(-1)^n x^n\left[-\frac{x}{(n+1)}+\frac{x^2}{(n+1)(n+2)}-\frac{x^3}{(n+1)(n+2)(n+3)}+\ldots\right]$$

$$\frac{n!}{(-1)^{n+1}x^n e^x}-\frac{n!}{(-1)^{n+1}x^n}\sum_{k=0}^{n}\frac{(-x)^k}{k!}=\frac{x}{(n+1)}-\frac{x^2}{(n+1)(n+2)}+\frac{x^3}{(n+1)(n+2)(n+3)}-\ldots$$

(EMCFF#2)

$$\frac{x}{(n+1)}-\frac{x^2}{(n+1)(n+2)}+\frac{x^3}{(n+1)(n+2)(n+3)}-\ldots=\cfrac{x}{n+1+\cfrac{(n+1)x}{n+2-x+\cfrac{(n+2)x}{n+3-x+\cfrac{(n+3)x}{n+4-x+\cfrac{(n+4)x}{\ddots}}}}}$$

$$\frac{n!}{(-1)^{n+1}x^n e^x}-\frac{n!}{(-1)^{n+1}x^n}\sum_{k=0}^{n}\frac{(-x)^k}{k!}=\cfrac{x}{n+1+\cfrac{(n+1)x}{n+2-x+\cfrac{(n+2)x}{n+3-x+\cfrac{(n+3)x}{n+4-x+\cfrac{(n+4)x}{\ddots}}}}}$$

$$n+1+\cfrac{(n+1)x}{n+2-x+\cfrac{(n+2)x}{n+3-x+\cfrac{(n+3)x}{n+4-x+\cfrac{(n+4)x}{\ddots}}}}=\frac{(-1)^{n+1}x^{n+1}}{\frac{n!}{e^x}-n!\sum_{k=0}^{n}\frac{(-x)^k}{k!}}$$

$$\boxed{n+1-x+\cfrac{(n+1)x}{n+2-x+\cfrac{(n+2)x}{n+3-x+\cfrac{(n+3)x}{n+4-x+\cfrac{(n+4)x}{\ddots}}}}=\frac{(-x)^{n+1}}{\frac{n!}{e^x}-n!\sum_{k=0}^{n}\frac{(-x)^k}{k!}}-x}$$

Exmples

Set $x=1$ and you will get:

$$n+\cfrac{n+1}{n+1+\cfrac{n+2}{n+2+\cfrac{n+3}{n+3+\cfrac{n+4}{\ddots}}}}=\frac{(-1)^{n+1}}{\frac{n!}{e}-\sum_{k=0}^{n}\frac{n!}{k!}(-1)^k}-1$$

Set $x=1$, $n=0$ and you will get:

$$0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{\ddots}}}}=\frac{(-1)^1}{\frac{1}{e}-\sum_{k=0}^{0}\frac{(-1)^k}{k!}}-1=\frac{-1}{\frac{1}{e}-1}-1=\frac{1}{e-1}$$

Set $x=1$, $n=1$ and you will get:

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{\ddots}}}}=\frac{1}{\frac{1}{e}-\sum_{k=0}^{1}\frac{(-1)^k}{k!}}-1=\frac{1}{\frac{1}{e}-0}-1=e-1$$

Set $x=n$ and you will get:

$$1+\cfrac{(n+1)n}{2+\cfrac{(n+2)n}{3+\cfrac{(n+3)n}{4+\cfrac{(n+4)n}{\ddots}}}}=\frac{(-n)^{n+1}}{\frac{n!}{e^n}-n!\sum_{k=0}^{n}\frac{(-n)^k}{k!}}-n$$

Set $x=n$, $n=3$ and you will get:

$$1+\cfrac{3\cdot 4}{2+\cfrac{3\cdot 5}{3+\cfrac{3\cdot 6}{4+\cfrac{3\cdot 7}{\ddots}}}}=\frac{(-3)^4}{\frac{6}{e^3}-6\sum_{k=0}^{3}\frac{(-3)^k}{k!}}-3=\frac{81}{\frac{6}{e^3}+12}-3=\frac{15e^3-6}{2+4e^3}=3.586049940664...$$

etc …

Formula No. 22

Replace $x \leftarrow -x$ in Formula No. 21 and you will get:

$$n+1+x-\cfrac{(n+1)x}{n+2+x-\cfrac{(n+2)x}{n+3+x-\cfrac{(n+3)x}{n+4+x-\cfrac{(n+4)x}{\ddots}}}}=\frac{x^{n+1}}{e^x n!-n!\sum_{k=0}^{n}\frac{x^k}{k!}}+x$$

Set $x=1$ and you will get:

$$n+2-\cfrac{n+1}{n+3-\cfrac{n+2}{n+4-\cfrac{n+3}{n+5-\cfrac{n+4}{\ddots}}}}=\frac{1}{en!-\sum_{k=0}^{n}\frac{n!}{k!}}+1$$

Set $x=1$, $n=0$ and you will get:

$$2-\cfrac{1}{3-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{\ddots}}}}=\frac{1}{e-\sum_{k=0}^{0}\frac{1}{k!}}+1=\frac{1}{e-1}+1$$

Set $x=n$ and you will get:

$$1+\cfrac{(n+1)n}{2+\cfrac{(n+2)n}{3+\cfrac{(n+3)n}{4+\cfrac{(n+4)n}{\ddots}}}}=1+\cfrac{n+1}{\frac{2}{n}+\cfrac{n+2}{3+\cfrac{n+3}{\frac{4}{n}+\cfrac{n+4}{5+\cfrac{n+5}{\frac{6}{n}+\cfrac{n+6}{7+\cfrac{n+7}{\ddots}}}}}}}=\frac{(-n)^{n+1}}{\frac{n!}{e^n}-n!\sum_{k=0}^{n}\frac{(-n)^k}{k!}}-n$$

Set $x=n$, $n=2$ and you will get:

$$1+\cfrac{3}{1+\cfrac{4}{3+\cfrac{5}{2+\cfrac{6}{5+\cfrac{7}{3+\cfrac{8}{\ddots}}}}}}=2\cdot\left(\frac{e^2+1}{e^2-1}\right)$$

Formula No. 23

Replace $x \leftarrow \frac{1}{x}$ in Formula No. 21 and you will get:

$$n+1+\frac{1}{x}-\cfrac{(n+1)\frac{1}{x}}{n+2+\frac{1}{x}-\cfrac{(n+2)\frac{1}{x}}{n+3+\frac{1}{x}-\cfrac{(n+3)\frac{1}{x}}{n+4+\frac{1}{x}-\cfrac{(n+4)\frac{1}{x}}{\ddots}}}}=\frac{1}{\sqrt[x]{e}\cdot x^{n+1}n!-x^{n+1}\sum_{k=0}^{n}\frac{n!}{x^k k!}}+\frac{1}{x}$$

$$\boxed{nx+1x+1-\cfrac{(n+1)x}{nx+2x+1-\cfrac{(n+2)x}{nx+3x+1-\cfrac{(n+3)x}{nx+4x+1-\cfrac{(n+4)x}{\ddots}}}}=\frac{1}{\sqrt[x]{e}\cdot x^{n}n!-\sum_{k=0}^{n}\frac{x^n n!}{x^k k!}}+1}$$

Exmples:

Set $x=n$ and you will get:

$$n^2+n+1-\cfrac{n(n+1)}{n^2+2n+1-\cfrac{n(n+2)}{n^2+3n+1-\cfrac{n(n+3)}{n^2+4n+1-\cfrac{n(n+4)}{\ddots}}}}=\frac{1}{\sqrt[n]{e}\cdot n^{n}n!-\sum_{k=0}^{n}\frac{n^n n!}{n^k k!}}+1$$

Formula No. 24

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$$

$$\frac{1}{e^x} = \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + (-1)^{n+1} \frac{x^{n+1}}{(n+1)!} + \ldots$$

$$e^x + \frac{1}{e^x} = \frac{2x^0}{0!} + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \ldots + \frac{2x^{2n}}{(2n)!} + \frac{2x^{2n+2}}{(2n+2)!} + \ldots$$

$$\frac{e^x}{2} + \frac{1}{2e^x} = \frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+2}}{(2n+2)!} + \ldots$$

$$(2n)!\left(\frac{e^x}{2} + \frac{1}{2e^x}\right) = \frac{(2n)!x^0}{0!} + \frac{(2n)!x^2}{2!} + \frac{(2n)!x^4}{4!} + \frac{(2n)!x^6}{6!} + \ldots + \frac{(2n)!x^{2n}}{(2n)!} + \frac{(2n)!x^{2n+2}}{(2n+2)!} + \ldots$$

$$(2n)!\left(\frac{e^x}{2} + \frac{1}{2e^x}\right) = \left[\frac{(2n)!x^0}{0!} + \frac{(2n)!x^2}{2!} + \frac{(2n)!x^4}{4!} + \ldots + \frac{(2n)!x^{2n}}{(2n)!}\right] + \left[\frac{(2n)!x^{2n+2}}{(2n+2)!} + \frac{(2n)!x^{2n+4}}{(2n+4)!} + \frac{(2n)!x^{2n+6}}{(2n+6)!} + \ldots\right]$$

$$(2n)!\left(\frac{e^x}{2} + \frac{1}{2e^x}\right) = (2n)!\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + x^{2n}\left[\frac{(2n)!x^2}{(2n+2)!} + \frac{(2n)!x^4}{(2n+4)!} + \frac{(2n)!x^6}{(2n+6)!} + \ldots\right]$$

$$\frac{(2n)!}{x^{2n}}\left(\frac{e^x}{2} + \frac{1}{2e^x} - \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}\right) = \frac{(2n)!x^2}{(2n+2)!} + \frac{(2n)!x^4}{(2n+4)!} + \frac{(2n)!x^6}{(2n+6)!} + \ldots$$

(EMCFF#1)

$$\frac{(2n)!}{x^{2n}}\left(\frac{e^x}{2} + \frac{1}{2e^x} - \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}\right) = \frac{x^2}{(2n+1)(2n+2)} + \frac{x^4}{(2n+1)(2n+2)(2n+3)(2n+4)} + \ldots =$$

$$\cfrac{x^2}{(2n+1)\cdot(2n+2) - \cfrac{(2n+1)\cdot(2n+2)\cdot x^2}{(2n+3)\cdot(2n+4) + x^2 - \cfrac{(2n+3)\cdot(2n+4)\cdot x^2}{(2n+5)\cdot(2n+6) + x^2 - \cfrac{(2n+5)\cdot(2n+6)\cdot x^2}{(2n+7)\cdot(2n+8) + x^2 - \cfrac{(2n+7)\cdot(2n+8)\cdot x^2}{\ddots}}}}}$$

$$\boxed{(2n+1)\cdot(2n+2) + x^2 - \cfrac{(2n+1)\cdot(2n+2)\cdot x^2}{(2n+3)\cdot(2n+4) + x^2 - \cfrac{(2n+3)\cdot(2n+4)\cdot x^2}{(2n+5)\cdot(2n+6) + x^2 - \cfrac{(2n+5)\cdot(2n+6)\cdot x^2}{\ddots}}} = \cfrac{x^2}{\cfrac{(2n)!}{x^{2n}}\left(\cfrac{e^x}{2} + \cfrac{1}{2e^x} - \sum_{k=0}^{n} \cfrac{x^{2k}}{(2k)!}\right)} + x^2}$$

Formula No. 25

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$$

$$\frac{1}{e^x} = \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + (-1)^{n+1} \frac{x^{n+1}}{(n+1)!} + \ldots$$

$$e^x - \frac{1}{e^x} = \frac{2x^1}{1!} + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \ldots + \frac{2x^{2n-1}}{(2n-1)!} + \frac{2x^{2n+1}}{(2n+1)!} + \ldots$$

$$\frac{e^x}{2} - \frac{1}{2e^x} = \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$(2n-1)!\left(\frac{e^x}{2} - \frac{1}{2e^x}\right) = \frac{(2n-1)!x^1}{1!} + \frac{(2n-1)!x^3}{3!} + \frac{(2n-1)!x^5}{5!} + \frac{(2n-1)!x^7}{7!} + \ldots + \frac{(2n-1)!x^{2n-1}}{(2n-1)!} + \frac{(2n-1)!x^{2n+1}}{(2n+1)!} + \ldots$$

$$(2n-1)!\left(\frac{e^x}{2} - \frac{1}{2e^x}\right) = \left[\frac{(2n-1)!x^1}{1!} + \frac{(2n-1)!x^3}{3!} + \frac{(2n-1)!x^5}{5!} + \frac{(2n-1)!x^7}{7!} + \ldots + \frac{(2n-1)!x^{2n-1}}{(2n-1)!}\right] + \left[\frac{(2n-1)!x^{2n+1}}{(2n+1)!} + \ldots\right]$$

$$(2n-1)!\left(\frac{e^x}{2} - \frac{1}{2e^x}\right) = (2n-1)!\sum_{k=0}^{n} \frac{x^{2k-1}}{(2k-1)!} + x^{2n-1}\left[\frac{(2n-1)!x^2}{(2n+1)!} + \frac{(2n-1)!x^4}{(2n+3)!} + \frac{(2n-1)!x^6}{(2n+5)!} + \ldots\right]$$

$$\frac{(2n-1)!}{x^{2n-1}}\left(\frac{e^x}{2} - \frac{1}{2e^x} - \sum_{k=0}^{n} \frac{x^{2k-1}}{(2k-1)!}\right) = \frac{(2n-1)!x^2}{(2n+1)!} + \frac{(2n-1)!x^4}{(2n+3)!} + \frac{(2n-1)!x^6}{(2n+5)!} + \ldots$$

$$\frac{(2n-1)!}{x^{2n-1}}\left(\frac{e^x}{2} - \frac{1}{2e^x} - \sum_{k=0}^{n} \frac{x^{2k-1}}{(2k-1)!}\right) = \frac{x^2}{(2n+0)(2n+1)} + \frac{x^4}{(2n+0)(2n+1)(2n+2)(2n+3)} + \ldots =$$

(EMCFF#1)

$$\cfrac{x^2}{(2n+0)\cdot(2n+1) - \cfrac{(2n+0)\cdot(2n+1)\cdot x^2}{(2n+2)\cdot(2n+3) + x^2 - \cfrac{(2n+2)\cdot(2n+3)\cdot x^2}{(2n+4)\cdot(2n+5) + x^2 - \cfrac{(2n+4)\cdot(2n+5)\cdot x^2}{(2n+6)\cdot(2n+7) + x^2 - \cfrac{(2n+6)\cdot(2n+7)\cdot x^2}{\ddots}}}}}$$

$$\boxed{(2n+0)\cdot(2n+1) + x^2 - \cfrac{(2n+0)\cdot(2n+1)\cdot x^2}{(2n+2)\cdot(2n+3) + x^2 - \cfrac{(2n+2)\cdot(2n+3)\cdot x^2}{(2n+4)\cdot(2n+5) + x^2 - \cfrac{(2n+4)\cdot(2n+5)\cdot x^2}{\ddots}}} = \cfrac{x^2}{\cfrac{(2n-1)!}{x^{2n-1}}\left(\cfrac{e^x}{2} - \cfrac{1}{2e^x} - \sum_{k=0}^{n} \cfrac{x^{2k-1}}{(2k-1)!}\right)} + x^2}$$

Formula No. 26

Set $2n+1=t$ in formula no. 24 and you will get:

$$(t)\cdot(t+1)+x^2-\cfrac{(t)\cdot(t+1)\cdot x^2}{(t+2)\cdot(t+3)+x^2-\cfrac{(t+2)\cdot(t+3)\cdot x^2}{(t+4)\cdot(t+5)+x^2-\cfrac{(t+4)\cdot(t+5)\cdot x^2}{\ddots}}}=\frac{x^2}{\frac{(t-1)!}{x^{t-1}}\left(\frac{e^x}{2}+\frac{1}{2e^x}-\sum_{k=0}^{(t-1)/2}\frac{x^{2k}}{(2k)!}\right)}+x^2$$

Set $2n=t$ in formula no. 25 and you will get:

$$(t)\cdot(t+1)+x^2-\cfrac{(t)\cdot(t+1)\cdot x^2}{(t+2)\cdot(t+3)+x^2-\cfrac{(t+2)\cdot(t+3)\cdot x^2}{(t+4)\cdot(t+5)+x^2-\cfrac{(t+4)\cdot(t+5)\cdot x^2}{\ddots}}}=\frac{x^2}{\frac{(t-1)!}{x^{t-1}}\left(\frac{e^x}{2}-\frac{1}{2e^x}-\sum_{k=0}^{t/2}\frac{x^{2k-1}}{(2k-1)!}\right)}+x^2$$

both are the same as:

$$(t)\cdot(t+1)+x^2-\cfrac{(t)\cdot(t+1)\cdot x^2}{(t+2)\cdot(t+3)+x^2-\cfrac{(t+2)\cdot(t+3)\cdot x^2}{(t+4)\cdot(t+5)+x^2-\cfrac{(t+4)\cdot(t+5)\cdot x^2}{\ddots}}}=\frac{x^2}{\frac{(t-1)!}{x^{t-1}}\left(\frac{e^x}{2}+\frac{1}{2e^x}-\sum_{k=0}^{t-1}\frac{x^{k}}{(k)!}\right)}+x^2$$

Set $t=n+1$ in the new formula above and you will get:

$$(n+1)\cdot(n+2)+x^2-\cfrac{(n+1)\cdot(n+2)\cdot x^2}{(n+3)\cdot(n+4)+x^2-\cfrac{(n+3)\cdot(n+4)\cdot x^2}{(n+5)\cdot(n+6)+x^2-\cfrac{(n+5)\cdot(n+6)\cdot x^2}{\ddots}}}=\frac{x^2}{\frac{n!}{x^{n}}\left(\frac{e^x}{2}+\frac{1}{2e^x}-\sum_{k=0}^{n}\frac{x^{k}}{(k)!}\right)}+x^2$$

Exmples:

Set $x\leftarrow\sqrt{x}$, $n=0$ and you will get:

$$1\cdot 2+x-\cfrac{1\cdot 2\cdot x}{3\cdot 4+x-\cfrac{3\cdot 4\cdot x}{5\cdot 6+x-\cfrac{5\cdot 6\cdot x}{\ddots}}}=\frac{x}{\left(\frac{e^{\sqrt{x}}}{2}+\frac{1}{2e^{\sqrt{x}}}-1\right)}+x$$

Formula No. 27

$$(x-y)^n = x^n - nx^{n-1}y^1 + \frac{n(n-1)}{2!}x^{n-2}y^2 - \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots$$

(EMCFF#2)

$$\frac{x^n}{1} - \frac{x^n}{1}\cdot\frac{ny}{1x} + \frac{x^n}{1}\cdot\frac{ny}{1x}\cdot\frac{(n-1)y}{2x} - \frac{x^n}{1}\cdot\frac{ny}{1x}\cdot\frac{(n-1)y}{2x}\cdot\frac{(n-2)y}{3x} + \ldots = \cfrac{x^n}{1-\cfrac{1\cdot ny}{1x+ny-\cfrac{1x\cdot(n-1)y}{2x+(n-1)y-\cfrac{2x\cdot(n-2)y}{3x+(n-2)y-\cfrac{3x\cdot(n-3)y}{\ddots}}}}}$$

$$(x-y)^n = \cfrac{x^n}{1-\cfrac{1\cdot ny}{1x+ny-\cfrac{1x\cdot(n-1)y}{2x+(n-1)y-\cfrac{2x\cdot(n-2)y}{3x+(n-2)y-\cfrac{3x\cdot(n-3)y}{\ddots}}}}}$$

$$1-\cfrac{ny}{1x+ny-\cfrac{1x\cdot(n-1)y}{2x+(n-1)y-\cfrac{2x\cdot(n-2)y}{3x+(n-2)y-\cfrac{3x\cdot(n-3)y}{\ddots}}}} = \frac{x^n}{(x-y)^n}$$

$$1-\frac{1}{(1-y/x)^n} = \cfrac{ny}{1x+ny-\cfrac{1x\cdot(n-1)y}{2x+(n-1)y-\cfrac{2x\cdot(n-2)y}{3x+(n-2)y-\cfrac{3x\cdot(n-3)y}{\ddots}}}}$$

$$\boxed{1x+(n-0)y-\cfrac{1x\cdot(n-1)y}{2x+(n-1)y-\cfrac{2x\cdot(n-2)y}{3x+(n-2)y-\cfrac{3x\cdot(n-3)y}{\ddots\;\cfrac{\ddots}{(n-1)x+2y-\cfrac{(n-1)x\cdot 1y}{(n-0)x+1y}}}}} = \cfrac{ny}{1-\cfrac{1}{(1-y/x)^n}}}$$

Formula No. 28

$$\sum_{m=k}^{n}\binom{m}{k}=\binom{n+1}{k+1} \qquad \text{hockey-stick identity}$$

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

$$\binom{n+1}{k+1}=\sum_{m=k}^{n}\binom{m}{k}=\frac{k!}{k!0!}+\frac{(k+1)!}{k!1!}+\frac{(k+2)!}{k!2!}+\frac{(k+3)!}{k!3!}+\ldots+\frac{(n)!}{k!(n-k)!}$$

(EMCFF#1)

$$\frac{k!}{k!0!}+\frac{k!}{k!0!}\cdot\frac{(k+1)}{1}+\frac{k!}{k!0!}\cdot\frac{(k+1)}{1}\cdot\frac{(k+2)}{2}+\frac{k!}{k!0!}\cdot\frac{(k+1)}{1}\cdot\frac{(k+2)}{2}\cdot\frac{(k+3)}{3}+\ldots+\frac{k!}{k!0!}\cdot\frac{(k+1)}{1}\cdots\frac{(n)}{(n-k)}=$$

$$\cfrac{k!}{k!0!-\cfrac{k!0!(k+1)}{1+(k+1)-\cfrac{1(k+2)}{2+(k+2)-\cfrac{2(k+3)}{3+(k+3)-\cfrac{\ddots}{\ddots\,\cfrac{(n-k-2)(n-1)}{(n-k-1)+(n-1)-\cfrac{(n-k-1)(n)}{(n-k)+(n)}}}}}}}$$

$$\binom{n+1}{k+1}=\cfrac{k!}{k!-\cfrac{k!(k+1)}{1+(k+1)-\cfrac{1(k+2)}{2+(k+2)-\cfrac{2(k+3)}{3+(k+3)-\cfrac{\ddots}{\ddots\,\cfrac{(n-k-2)(n-1)}{(n-k-1)+(n-1)-\cfrac{(n-k-1)(n)}{(n-k)+(n)}}}}}}}$$

$$\binom{n+1}{k+1}=\cfrac{1}{1-\cfrac{(k+1)}{1+(k+1)-\cfrac{1(k+1)}{2+(k+2)-\cfrac{2(k+2)}{3+(k+3)-\cfrac{\ddots}{\ddots\,\cfrac{(n-k-2)(n-1)}{(n-k-1)+(n-1)-\cfrac{(n-k-1)(n)}{(n-k)+(n)}}}}}}}$$

$$1-\cfrac{(k+1)}{1+(k+1)-\cfrac{1(k+2)}{2+(k+2)-\cfrac{2(k+3)}{3+(k+3)-\cfrac{\ddots}{\ddots\;\cfrac{(n-k-2)(n-1)}{(n-k-1)+(n-1)-\cfrac{(n-k-1)(n)}{(n-k)+(n)}}}}}}=\cfrac{1}{\binom{n+1}{k+1}}$$

$$1-\cfrac{1}{\binom{n+1}{k+1}}=\cfrac{(k+1)}{1+(k+1)-\cfrac{1(k+2)}{2+(k+2)-\cfrac{2(k+3)}{3+(k+3)-\cfrac{\ddots}{\ddots\;\cfrac{(n-k-2)(n-1)}{(n-k-1)+(n-1)-\cfrac{(n-k-1)(n)}{(n-k)+(n)}}}}}}$$

$$\boxed{k+2-\cfrac{1(k+2)}{k+4-\cfrac{2(k+3)}{k+6-\cfrac{\ddots}{\ddots\;\cfrac{(n-k-2)(k+(n-k-1))}{k+2(n-k-1)-\cfrac{(n-k-1)(k+(n-k))}{k+2(n-k)}}}}}=\cfrac{k+1}{1-\cfrac{1}{\binom{n+1}{k+1}}}}$$

Exmples:

Replace $k \leftarrow k-1$ and you will get:

$$\boxed{k+1-\cfrac{1(k+1)}{k+3-\cfrac{2(k+2)}{k+5-\cfrac{\ddots}{\ddots\;\cfrac{(n-k-1)(k+(n-k-1))}{k+2(n-k)-1-\cfrac{(n-k)(k+(n-k))}{k+2(n-k+1)-1}}}}}=\cfrac{k}{1-\cfrac{1}{\binom{n+1}{k}}}}$$

Set $k=0$ and you will get:

$$2-\cfrac{1\cdot 2}{4-\cfrac{2\cdot 3}{6-\cfrac{3\cdot 4}{\ddots\;\cfrac{\ddots}{2(n-2)-\cfrac{(n-2)\cdot(n-1)}{2(n-1)-\cfrac{(n-1)\cdot n}{2n}}}}}}=\cfrac{1}{1-\cfrac{1}{n+1}}=\cfrac{n+1}{n}=1+\cfrac{1}{n}$$

Formula No. 29

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n$$ Central binomial coefficient

$$\binom{2n}{n}=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{n!n!}$$

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n=\frac{0!x^0}{0!^2}+\frac{2!x^1}{1!^2}+\frac{4!x^2}{2!^2}+\frac{6!x^3}{3!^2}+\frac{8!x^4}{4!^2}$$

$$\frac{0!x^0}{0!^2}+\frac{0!x^0}{0!^2}\cdot\frac{1\cdot2\cdot x}{1\cdot1}+\frac{0!x^0}{0!^2}\cdot\frac{1\cdot2\cdot x}{1\cdot1}\cdot\frac{3\cdot4\cdot x}{2\cdot2}+\frac{0!x^0}{0!^2}\cdot\frac{1\cdot2\cdot x}{1\cdot1}\cdot\frac{3\cdot4\cdot x}{2\cdot2}\cdot\frac{5\cdot6\cdot x}{3\cdot3}+\ldots=$$

$$\cfrac{0!x^0}{0!^2-\cfrac{0!^2\cdot1\cdot2\cdot x}{1\cdot1+1\cdot2\cdot x-\cfrac{1\cdot1\cdot3\cdot4\cdot x}{2\cdot2+3\cdot4\cdot x-\cfrac{2\cdot2\cdot5\cdot6\cdot x}{3\cdot3+5\cdot6\cdot x-\cfrac{3\cdot3\cdot7\cdot8\cdot x}{\ddots}}}}}$$

$$\frac{1}{\sqrt{1-4x}}=\cfrac{1}{1-\cfrac{1\cdot2\cdot x}{1^2+1\cdot2\cdot x-\cfrac{1^2\cdot3\cdot4\cdot x}{2^2+3\cdot4\cdot x-\cfrac{2^2\cdot5\cdot6\cdot x}{3^2+5\cdot6\cdot x-\cfrac{3^2\cdot7\cdot8\cdot x}{\ddots}}}}}$$

$$1-\cfrac{1\cdot2\cdot x}{1^2+1\cdot2\cdot x-\cfrac{1^2\cdot3\cdot4\cdot x}{2^2+3\cdot4\cdot x-\cfrac{2^2\cdot5\cdot6\cdot x}{3^2+5\cdot6\cdot x-\cfrac{3^2\cdot7\cdot8\cdot x}{\ddots}}}}=\sqrt{1-4x}$$

$$\boxed{1^2+1\cdot2\cdot x-\cfrac{1^2\cdot3\cdot4\cdot x}{2^2+3\cdot4\cdot x-\cfrac{2^2\cdot5\cdot6\cdot x}{3^2+5\cdot6\cdot x-\cfrac{3^2\cdot7\cdot8\cdot x}{\ddots}}}=\frac{2x}{1-\sqrt{1-4x}}}$$

$x<1/4$

Formula No. 30

$$\sqrt[n]{x+y}=(x+y)^{1/n}=\sum_{k=0}^{\infty}x^k y^{1/n-k}\binom{1/n}{k}$$

$$\sqrt[n]{x+y}=\sqrt[n]{y}+\frac{x^1\sqrt[n]{y}}{1!n^1y^1}-\frac{(n-1)}{2!n^2}\cdot\frac{x^2\sqrt[n]{y}}{y^2}+\frac{(n-1)(2n-1)}{3!n^3}\cdot\frac{x^3\sqrt[n]{y}}{y^3}-\frac{(n-1)(2n-1)(3n-1)}{4!n^4}\cdot\frac{x^4\sqrt[n]{y}}{y^4}+$$

$$\frac{(n-1)(2n-1)(3n-1)(4n-1)}{5!n^5}\cdot\frac{x^5\sqrt[n]{y}}{y^5}-\frac{(n-1)(2n-1)(3n-1)(4n-1)(5n-1)}{6!n^6}\cdot\frac{x^6\sqrt[n]{y}}{y^6}+O(x^7)$$

$$\frac{\sqrt[n]{x+y}-\sqrt[n]{y}}{\sqrt[n]{y}}=\frac{x^1}{1!n^1y^1}-\frac{(n-1)}{2!n^2}\cdot\frac{x^2}{y^2}+\frac{(n-1)(2n-1)}{3!n^3}\cdot\frac{x^3}{y^3}-\frac{(n-1)(2n-1)(3n-1)}{4!n^4}\cdot\frac{x^4}{y^4}+$$

$$\frac{(n-1)(2n-1)(3n-1)(4n-1)}{5!n^5}\cdot\frac{x^5}{y^5}-\frac{(n-1)(2n-1)(3n-1)(4n-1)(5n-1)}{6!n^6}\cdot\frac{x^6}{y^6}+\dots$$

$$\frac{\sqrt[n]{x+y}-\sqrt[n]{y}}{\sqrt[n]{y}}=\frac{1\cdot x}{1\cdot ny}-\left(\frac{1\cdot x}{1\cdot ny}\cdot\frac{(n-1)\cdot x}{2\cdot ny}\right)+\left(\frac{1\cdot x}{1\cdot ny}\cdot\frac{(n-1)\cdot x}{2\cdot ny}\cdot\frac{(2n-1)\cdot x}{3\cdot ny}\right)-\left(\frac{1\cdot x}{1\cdot ny}\cdot\frac{(n-1)\cdot x}{2\cdot ny}\cdot\frac{(2n-1)\cdot x}{3\cdot ny}\cdot\frac{(3n-1)\cdot x}{4\cdot ny}\right)+$$

$$\left(\frac{1\cdot x}{1\cdot ny}\cdot\frac{(n-1)\cdot x}{2\cdot ny}\cdot\frac{(2n-1)\cdot x}{3\cdot ny}\cdot\frac{(3n-1)\cdot x}{4\cdot ny}\cdot\frac{(4n-1)\cdot x}{5\cdot ny}\right)-\left(\frac{1\cdot x}{1\cdot ny}\cdot\frac{(n-1)\cdot x}{2\cdot ny}\cdot\frac{(2n-1)\cdot x}{3\cdot ny}\cdot\frac{(3n-1)\cdot x}{4\cdot ny}\cdot\frac{(4n-1)\cdot x}{5\cdot ny}\cdot\frac{(5n-1)\cdot x}{6\cdot ny}\right)+\dots$$

$$\frac{\sqrt[n]{x+y}-\sqrt[n]{y}}{\sqrt[n]{y}}=\cfrac{1x}{1ny+\cfrac{1ny\cdot(n-1)\cdot x}{2ny-(n-1)\cdot x+\cfrac{2ny\cdot(2n-1)\cdot x}{3ny-(2n-1)\cdot x+\cfrac{3ny\cdot(3n-1)\cdot x}{4ny-(3n-1)\cdot x+\cfrac{4ny\cdot(4n-1)\cdot x}{5ny-(4n-1)\cdot x+\cfrac{5ny\cdot(5n-1)\cdot x}{\ddots}}}}}}$$

$$1ny+\cfrac{1ny\cdot(1n-1)\cdot x}{2ny-(1n-1)\cdot x+\cfrac{2ny\cdot(2n-1)\cdot x}{3ny-(2n-1)\cdot x+\cfrac{3ny\cdot(3n-1)\cdot x}{4ny-(3n-1)\cdot x+\cfrac{4ny\cdot(4n-1)\cdot x}{\ddots}}}}=\frac{x\sqrt[n]{y}}{\sqrt[n]{x+y}-\sqrt[n]{y}}$$

$$\boxed{1ny-(0n-1)\cdot x+\cfrac{1ny\cdot(1n-1)\cdot x}{2ny-(1n-1)\cdot x+\cfrac{2ny\cdot(2n-1)\cdot x}{3ny-(2n-1)\cdot x+\cfrac{3ny\cdot(3n-1)\cdot x}{4ny-(3n-1)\cdot x+\cfrac{4ny\cdot(4n-1)\cdot x}{\ddots}}}}=\frac{x\sqrt[n]{y}}{\sqrt[n]{x+y}-\sqrt[n]{y}}+x}$$

Set $x=-y$ and you will get:

$$1ny+(0n-1)\cdot y-\cfrac{1ny\cdot(1n-1)\cdot y}{2ny+(1n-1)\cdot y-\cfrac{2ny\cdot(2n-1)\cdot y}{3ny+(2n-1)\cdot y-\cfrac{3ny\cdot(3n-1)\cdot y}{4ny+(3n-1)\cdot y-\cfrac{4ny\cdot(4n-1)\cdot y}{\ddots}}}}=0$$

Set $x=1$, $y=\frac{x}{n}$ and you will get:

$$1x-(0n-1)+\cfrac{1x\cdot(1n-1)}{2x-(1n-1)+\cfrac{2x\cdot(2n-1)}{3x-(2n-1)+\cfrac{3x\cdot(3n-1)}{4x-(3n-1)+\cfrac{4x\cdot(4n-1)}{\ddots}}}}=\frac{1}{\sqrt[n]{\frac{n}{x}+1}-1}+1$$

Set $x=-y$, y=1 and you will get:

$$n-1-\cfrac{n\cdot(n-1)}{3n-1-\cfrac{2n\cdot(2n-1)}{5n-1-\cfrac{3n\cdot(3n-1)}{7n-1-\cfrac{4n\cdot(4n-1)}{\ddots}}}}=0$$

etc…