Riemann Hypothesis

Direct demonstration proposal

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I. Abstract:

In his 1859 article "On the number of prime numbers less than a given quantity", Bernhard Riemann formulated the hypothesis that all non-trivial zeros of the Zeta function have the real part 1/2.

This assertion, known as the "Riemann Hypothesis", remains unproven to this day.

The present paper is an attempt at a direct demonstration.

II. Demonstration :

The demonstration proposed here is based on two well-known results:

1. Zeta function as Hadamard product on one side:

$$\zeta(s) = \frac{e^{\left(\ln(2\pi)-1-\frac{\gamma}{2}\right)s}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho} \left(1-\frac{s}{\rho}\right) e^{s/\rho} \quad (1)$$

Where the ρ are the non trivial zeros of the Zeta function.

2. The value of ζ(-1) on the other hand:

$$\zeta(-1) = -\frac{1}{12} \quad (2)$$

Replacing s by -1 in expression (1) and equating (1) and (2), we obtain :

$$\zeta(-1) = \frac{e^{-(\ln(2\pi)-1-\frac{\gamma}{2})}}{2(-2)\sqrt{\pi}} \prod_{\rho} \left(1+\frac{1}{\rho}\right) e^{-1/\rho} = -\frac{1}{12}$$

Now, the calculation shows that

$$\frac{e^{-(\ln(2\pi)-1-\frac{\gamma}{2})}}{2(-2)\sqrt{\pi}} \approx -\frac{1}{12}$$

And so

$$\prod_{\rho} \left(1+\frac{1}{\rho}\right) e^{-1/\rho} = 1 \quad (3)$$

Since the number of zeros is infinite, the number of factors $\left(1+\frac{1}{\rho}\right)e^{-1/\rho}$ is also infinite, so that equality (3) can only be verified if each factor $\left(1+\frac{1}{\rho}\right)e^{-1/\rho}$ is equal to 1 and therefore for any zero ρ of the Zeta function :

$$\left(1+\frac{1}{\rho}\right)e^{-1/\rho}=1 \quad (4)$$

NB: This equality has been verified numerically for the first 100 zeros of the zeta function

Posing $\rho=\sigma+it$, then:

$$\left(1-\frac{\sigma-it}{\sigma^2+t^2}\right)e^{\frac{-\sigma}{\sigma^2+t^2}}(cos\frac{t}{\sigma^2+t^2}+isin\frac{t}{\sigma^2+t^2})=1 \qquad (5)$$

Now, $e^{\frac{-\sigma}{\sigma^2+t^2}}\approx 1$ because $0<\sigma<1$ (located inside the critical strip) and therefore $-\sigma \ll \sigma^2+t^2$ when t^2 tends to infinity.

So $\left(1-\frac{\sigma-it}{\sigma^2+t^2}\right)(cos\frac{t}{\sigma^2+t^2}+isin\frac{t}{\sigma^2+t^2})=1$

$$\Rightarrow \left[\left(1-\frac{\sigma}{\sigma^2+t^2}\right)+i\frac{t}{\sigma^2+t^2}\right](cos\frac{t}{\sigma^2+t^2}+isin\frac{t}{\sigma^2+t^2})=1$$

$$\Rightarrow \left[\left(1-\frac{\sigma}{\sigma^2+t^2}\right)\left(cos\frac{t}{\sigma^2+t^2}\right)-\left(\frac{t}{\sigma^2+t^2}\right)\left(sin\frac{t}{\sigma^2+t^2}\right)\right]+i\left[\left(1-\frac{\sigma}{\sigma^2+t^2}\right)\left(sin\frac{t}{\sigma^2+t^2}\right)+\left(\frac{t}{\sigma^2+t^2}\right)\left(cos\frac{t}{\sigma^2+t^2}\right)\right]=1$$

Equalizing the real and imaginary parts, we obtain the equations :

$$\left[\left(1-\frac{\sigma}{\sigma^2+t^2}\right)\left(cos\frac{t}{\sigma^2+t^2}\right)-\left(\frac{t}{\sigma^2+t^2}\right)\left(sin\frac{t}{\sigma^2+t^2}\right)\right]=1 \qquad (6)$$

and

$$\left[\left(1-\frac{\sigma}{\sigma^2+t^2}\right)\left(sin\frac{t}{\sigma^2+t^2}\right)+\left(\frac{t}{\sigma^2+t^2}\right)\left(cos\frac{t}{\sigma^2+t^2}\right)\right]=0 \qquad (7)$$

Squaring (6) and (7) gives :

$$\left(1-\frac{\sigma}{\sigma^2+t^2}\right)^2\left(cos\frac{t}{\sigma^2+t^2}\right)^2-2\left(1-\frac{\sigma}{\sigma^2+t^2}\right)\left(cos\frac{t}{\sigma^2+t^2}\right)\left(\frac{t}{\sigma^2+t^2}\right)\left(sin\frac{t}{\sigma^2+t^2}\right)+\left(\frac{t}{\sigma^2+t^2}\right)^2\left(sin\frac{t}{\sigma^2+t^2}\right)^2=1 \qquad (8)$$

and

$$\left(1-\frac{\sigma}{\sigma^2+t^2}\right)^2\left(sin\frac{t}{\sigma^2+t^2}\right)^2+2\left(1-\frac{\sigma}{\sigma^2+t^2}\right)\left(cos\frac{t}{\sigma^2+t^2}\right)\left(\frac{t}{\sigma^2+t^2}\right)\left(sin\frac{t}{\sigma^2+t^2}\right)+\left(\frac{t}{\sigma^2+t^2}\right)^2\left(cos\frac{t}{\sigma^2+t^2}\right)^2=0 \qquad (9)$$

Summing (8) and (9), it remains $\left(1 - \frac{\sigma}{\sigma^2+t^2}\right)^2 + \left(\frac{t}{\sigma^2+t^2}\right)^2 = 1$

$$\Rightarrow 1 - 2\frac{\sigma}{\sigma^2+t^2} + \frac{\sigma^2}{(\sigma^2+t^2)^2} + \frac{t^2}{(\sigma^2+t^2)^2} = 1$$

$$\Rightarrow \frac{\sigma^2}{(\sigma^2+t^2)^2} + \frac{t^2}{(\sigma^2+t^2)^2} = 2\frac{\sigma}{\sigma^2+t^2}$$

$$\Rightarrow \frac{\sigma^2+t^2}{(\sigma^2+t^2)^2} = 2\frac{\sigma}{\sigma^2+t^2}$$

$\Rightarrow 1 = 2\,\sigma$ and therefore

$$\boldsymbol{\sigma = \frac{1}{2}}$$

Vincent KOCH, November 24th 2023

III. Bibliography and videography

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   - The Basel Problem Part 1: Euler-Maclaurin Approximation
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   - But what is the Riemann zeta function? Visualizing analytic continuation
     https://www.youtube.com/watch?v=sD0NjbwqlYw&t=1166s