Application of Shoshany-Snodgrass Analysis to the Natario Warp Drive Spacetime with Zero Expansion

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Abstract

Alcubierre and Natario developed their warp drives spacetimes using the Arnowitt-Dresner-Misner ADM formalism considering the lapse function α always equal to 1.Recently Barak Shoshany and Ben Snodgrass considered the possibility of warp drive spacetimes in which the lapse function α is different than 1 in very special geometric cases and we arrive at very interesting results:In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time in months not in years a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth which is "only" proportional to the factor 10^{24}!!.The lapse function allows more effectively the negative energy density requirements when a ship travels with a speed of 200 times faster than light using the Shoshany-Snodgrass analysis. We reproduce here the Shoshany-Snodgrass analysis to the Natario warp drive spacetime with zero expansion.

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1 Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8],[9],[10] and mainly [18]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7],[8],[18] and mainly [32]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7],[8] and mainly [31]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8],[9],[10],[18],[31],[32]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

[1] do not violates Relativity

Alcubierre([12]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([11]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [12], see also eq 1 pg 3 in [1])[2][3] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.

The Natario warp drive equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg $[507(b)]$ $[534(a)]$ in [11])

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.[4]

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11]
with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-,+,+,+)$ we get the original matrices of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{2}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{3}$$

Alcubierre and Natario developed their warp drives considering the lapse function α always equal to 1.

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the Natario vector we have the following equation for the Natario warp drive with zero expansion:

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{4}$$

Recently Barak Shoshany and Ben Snodgrass in [34] considered the possibility of warp drive spacetimes in which the lapse function α is different than 1 in very special geometric cases.The equation for the Natario warp drive with zero expansion using a lapse function is given by::

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{5}$$

In this work we present the analysis of Barak Shoshany and Ben Snodgrass in [34] adapted to the Natario warp drive with zero expansion using a lapse function and we demonstrate how we arrived at the equation above.

[2]see Appendix E
[3]see the Remarks section on our system to quote pages in bibliographic references
[4]see again Appendix E

For the study of the original ADM formalism we use the approaches of MTW([11]) and Alcubierre([12]) and we adopt the Alcubierre convention for notation of equations and scripts.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes and the International System of units for energetic purposes.

This work is organized as follows:

- Section 2)-Introduces the Natario warp drive continuous shape function able to low the negative energy density requirements when a ship travels with a speed of 200 times faster than light.
  The negative energy density for such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!.

- Section 3)-Introduces the Natario warp drive continuous lapse function able to more effectively low the negative energy density requirements when a ship travels with a speed of 200 times faster than light using the Shoshany-Snodgrass analysis in [34].
  The lapse function possesses the following values:
  - inside the Natario warp bubble where the spaceship resides (flat spacetime) the lapse function is always 1.
  - outside the Natario warp bubble where an external observer resides watching the bubble passing by (also flat spacetime) the lapse function is again always 1.
  - in the Natario warped region where the derivatives of the Natario shape function are not null (curved spacetime) the lapse function possesses very large values.

- Section 4)-presents the original equation for the Natario warp drive spacetime with zero expansion and a constant velocity vs without lapse function or with a lapse function always equal to 1 in the original $3+1$ ADM formalism in a rigorous mathematical fashion. We recommend the study of the Appendices A and E at the end of the work in order to fully understand the mathematical demonstrations.

- Section 5)-presents the original equation for the Natario warp drive spacetime with zero expansion and a constant velocity vs with a large lapse function in the Natario warped region in the original $3+1$ ADM formalism in a rigorous mathematical fashion. We recommend the study of the Appendices I and J at the end of the work in order to fully understand the mathematical demonstrations.

- Section 6)-compares all these equations outlining differences and resemblances.

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating(see preface of second edition and pg $[337(a)], [337)b)]$ in [13]) and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pgs $[106(a)], [98(b)]$ in [23] and pgs $[394(a)][377(b)]$ in [24] pgs $[119(a)][226(b)]$ in [28])(see pg 37 in [35]).

It is very important to note that if a galaxy in the other side of the Universe at a billion light-years of distance outside the range of our Particle Horizon is moving away from us at a faster than light speed then superluminal velocities may well exists in Nature.So the warp drive may not be impossible at all.Natario also points out exactly this.(see pgs [10] and [11] in [29]).

What Alcubierre and Natario did was an attempt to reproduce the expansion of the Universe in a local way creating a local spacetime distortion that expands the spacetime behind a ship and contracts spacetime in front reproducing the superluminal expansion of the Universe moving away the departure point in an expansion and bringing together the destination point in a contraction.The expansion-contraction can be seen in the abs of the original Alcubierre paper in [1].Although Natario says in the abs of his paper in [2] that the expansion-contraction does not occurs in its spacetime in pg 5 of the Natario paper we can see the expansion-contraction occurring however the expansion of the normal volume elements or the trace of the extrinsic curvature is zero because the contraction in the radial direction is exactly balanced by the expansion in the perpendicular directions.

An excellent explanation on how a spacetime distortion or a perturbation pushes away a spaceship from the departure point and brings the ship to the destination point at faster than light speed can be seen at pg 34 in [21],pgs $[260(a)260(b)][261(a)261(b)]$ in [22].Note that in these works it can be seen that the perturbation do not obeys the time dilatation of the Lorentz transformations hence the speed limit of Special Relativity cannot be applied here.

An accelerated warp drive can only exists if the astronaut in the center of the warp bubble can somehow communicate with the warp bubble walls sending instructions to change its speed.But for signals at light speed the Horizon exists so light speed cannot be used to send signals to the front of the bubble.(see pg 16 in [7] and pg 21 in [8]).Besides in the Natario warp drive the negative energy density covers the entire bubble.(see pg 52 in [7] and pg 51 in [8]).Since the negative energy density have repulsive gravitational behavior the photon of light if possible to reach the bubble walls would then be deflected by the repulsive behavior never reaching the bubble walls(see pg $[116(a)][116(b)]$ in [26])

The solution that allows contact with the bubble walls was presented in pg 28 in [7] and pg 31 in [8]. Although the light cone of the external part of the warp bubble is causally disconnected from the astronaut who lies inside the large bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro bubble so these bubbles can be "engineered" to be sent to the large bubble. This idea seems to be endorsed by pg 34 in [21],pgs $[268(a)268(b)]$ in [22] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.See also an important point of view in [31].

This work covers only the lapse function for the Natario warp drive with zero expansion at a constant velocity.

But for the case of the warp drive equations with variable velocities vs (see [20],[30] and [33]) we still do not have geometries possessing a lapse function α.This(perhaps) will be done in future works.

Although this work was written to be independent self-contained and self-consistent it must be regarded as a companion work to our works in [16],[17],[18],[20],[30],[31],[32] and in [33].

2 The Natario warp drive continuous shape function

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can construct the Natario shape function $n(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[5].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{6}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{7}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function can be defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{8}$$

$$n(rs) = \frac{1}{2}[1 - [\frac{1}{2}[1 - tanh[@(rs - R)]]]] \tag{9}$$

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

[5] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

Another Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{10}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{11}$$

The shape function above also gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that like in the previous case the Alcubierre shape function is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.

For the second Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

- Numerical plot for the second shape function with $@ = 50000$ $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99970000000E+001$	1	0	$2,650396620740E-251$	0
$9,99980000000E+001$	1	0	$1,915169647489E-164$	0
$9,99990000000E+001$	1	0	$1,383896564748E-077$	0
$1,00000000000E+002$	0,5	0,5	$6,250000000000E+008$	$3,872591914849E-103$
$1,00001000000E+002$	0	0,5	$1,383896486082E-077$	0
$1,00002000000E+002$	0	0,5	$1,915169538624E-164$	0
$1,00003000000E+002$	0	0,5	$2,650396470082E-251$	0

- Numerical plot for the second shape function with $@ = 75000$ $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99980000000E+001$	1	0	$5,963392481410E-251$	0
$9,99990000000E+001$	1	0	$1,158345097767E-120$	0
$1,00000000000E+002$	0,5	0,5	$1,406250000000E+009$	$8,713331808411E-103$
$1,00001000000E+002$	0	0,5	$1,158344999000E-120$	0
$1,00002000000E+002$	0	0,5	$5,963391972940E-251$	0

- Numerical plot for the second shape function with $@ = 100000$ $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99990000000E+001$	1	0	$7,660678807684E-164$	0
$1,00000000000E+002$	0,5	0,5	$2,500000000000E+009$	$1,549036765940E-102$
$1,00001000000E+002$	0	0,5	$7,660677936765E-164$	0

The plots in the previous page demonstrate the important role of the thickness parameter @ in the warp bubble geometry wether in both Alcubierre or Natario warp drive spacetimes.For a bubble of 100 meters radius $R = 100$ the regions where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$(Natario warped region) becomes thicker or thinner as @ becomes higher.

Then the geometric position where both Alcubierre and Natario warped regions begins with respect to R the bubble radius is $rs = R - \epsilon < R$ and the geometric position where both Alcubierre and Natario warped regions ends with respect to R the bubble radius is $rs = R + \epsilon > R$

As large as @ becomes as smaller ϵ becomes too.

Note from the plots of the previous page that we really have two warped regions:

- 1)-The geometrized warped region where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$ (Natario warped region).
- 2)-The energized warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero.

The parameter @ affects both energized warped regions wether in Alcubierre or Natario cases but is more visible for the Alcubierre shape function because the warp factor WF in the Natario shape functions squeezes the energized warped region into a very small thickness.

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{12}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression(see Appendix G):

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{13}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for(see Appendix D):

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{14}$$

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{15}$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[6] of about $6 \times 10^{24} kg$

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the square derivative of the Natario shape function possesses values of 10^{-102} ameliorating the factor 10^{48} making the warp drive negative energy density more "affordable". For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

[6]see Wikipedia:The free Encyclopedia

3 The Natario warp drive continuous lapse function

In the previous section we presented the Natario warp drive continuous shape function defined using the equation of the Alcubierre shape function.[7].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \quad (16)$$

The Natario warp drive continuous shape function is defined as being:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \quad (17)$$

The expression for the first order derivative of the Natario shape function is :

$$n'(rs) = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f'(rs) \quad (18)$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \quad (19)$$

With the first order derivative of the Alcubierre shape function being:

$$f'(rs) = -\frac{1}{2}[\frac{@}{cosh^2[@(rs - R)]}] \quad (20)$$

The term WF in the Natario shape function is dimensionless:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

[7] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

We introduce here the Natario warp drive continuous lapse function with a value equal to 1 in the regions inside and outside the Natario bubble while having large values in the Natario warped region.The Natario warp drive continuous lapse function is given by;

$$a(rs) = (\frac{1}{2}[1 - (tanh[@(rs - R)])^{(2)}])^{(-WF)} \tag{21}$$

$$a(rs) = (\frac{1}{2}[1 - (tanh[@(rs - R)])^{(2)}])^{(\frac{1}{WF})} \tag{22}$$

Its square is then given by:(see Appendix I for details)

$$a(rs)^2 = (\frac{1}{2}[1 - (tanh[@(rs - R)])^{(2)}])^{(\frac{1}{2WF})} \tag{23}$$

rs	$a(rs)$	$a(rs)^2$	$n'(rs)$
$9,999500000000E+00$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$
$9,999600000000E+00$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$
$9,999700000000E+00$	$1,000000000037E+000$	$1,00000000007487E+000$	$0,00000000000E+000$
$9,999800000000E+00$	$1,000000824462E+000$	$1,00000164892427E+000$	$0,00000000000E+000$
$9,999900000000E+00$	$1,018325027378E+000$	$1,03698586138517E+000$	$0,00000000000E+000$
$1,000000000000E+01$	$1,606938044259E+060$	$2,58224987808691E+120$	$6,22301527786E-052$
$1,000010000000E+01$	$1,018325027273E+000$	$1,03698586117111E+000$	$0,00000000000E+000$
$1,000020000000E+01$	$1,000000824462E+000$	$1,00000164892427E+000$	$0,00000000000E+000$
$1,000030000000E+01$	$1,000000000037E+000$	$1,00000000007487E+000$	$0,00000000000E+000$
$1,000040000000E+01$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$
$1,000050000000E+01$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$

rs	$a(rs)$	$a(rs)^2$	$n'(rs)^2$
$9,999500000000E+00$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$
$9,999600000000E+00$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$
$9,999700000000E+00$	$1,000000000037E+000$	$1,00000000007487E+000$	$0,00000000000E+000$
$9,999800000000E+00$	$1,000000824462E+000$	$1,00000164892427E+000$	$0,00000000000E+000$
$9,999900000000E+00$	$1,018325027378E+000$	$1,03698586138517E+000$	$0,00000000000E+000$
$1,000000000000E+01$	$1,606938044259E+060$	$2,58224987808691E+120$	$3,8725919148493E-103$
$1,000010000000E+01$	$1,018325027273E+000$	$1,03698586117111E+000$	$0,00000000000E+000$
$1,000020000000E+01$	$1,000000824462E+000$	$1,00000164892427E+000$	$0,00000000000E+000$
$1,000030000000E+01$	$1,000000000037E+000$	$1,00000000007487E+000$	$0,00000000000E+000$
$1,000040000000E+01$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$
$1,000050000000E+01$	$1,000000000000E+000$	$1,00000000000000E+000$	$0,00000000000E+000$

Above are presented the numerical plots for the Natario warp drive continuous lapse function its square and the first order derivative and also its square of the Natario warp drive continuous shape function with $@ = 50000$ $R = 100$ meters and warp factor with a value $WF = 200$.[8].The values of the Natario warp drive continuous lapse function and its square are equal to 1 inside and outside the Natario bubble while having large values in the Natario warped region.The derivative of the Natario warp drive continuous shape function and its square have very low values in the Natario warped region.

[8]compare these plots with the plots of the previous section

The extrinsic curvatures are essential to calculate the negative energy density requirements.Below are presented the extrinsic curvatures for the Natario warp drive without a lapse function or with a lapse function always equal to 1.see pg 5 in [2].see also Appendix B for details.

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(rs)\cos\theta \tag{24}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(rs)\cos\theta \tag{25}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(rs)\cos\theta \tag{26}$$

$$K_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\theta}{r}\right) + \frac{1}{r}\frac{\partial X^r}{\partial\theta}\right] = v_s\sin\theta\left(n'(rs) + \frac{r}{2}n''(rs)\right) \tag{27}$$

$$K_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\varphi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial X^r}{\partial\varphi}\right] = 0 \tag{28}$$

$$K_{\theta\varphi} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{X^\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial X^\theta}{\partial\varphi}\right] = 0 \tag{29}$$

The expansion of the normal volume elements is zero and given by:

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{30}$$

Below are presented the extrinsic curvatures for the Natario warp drive with a lapse function that is 1 in the regions inside and outside the Natario bubble while being large in the Natario warped region.See Appendix I for details.[9]

$$K_{rr} = (\frac{1}{a(rs)})(\frac{\partial X^r}{\partial r}) = -2(\frac{1}{a(rs)})v_s n'(rs)\cos\theta \tag{31}$$

$$K_{\theta\theta} = (\frac{1}{a(rs)})(\frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r}) = (\frac{1}{a(rs)})v_s n'(rs)\cos\theta \tag{32}$$

$$K_{\varphi\varphi} = (\frac{1}{a(rs)})(\frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r}) = (\frac{1}{a(rs)})v_s n'(rs)\cos\theta \tag{33}$$

$$K_{r\theta} = (\frac{1}{a(rs)})\frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\theta}{r}\right) + \frac{1}{r}\frac{\partial X^r}{\partial\theta}\right] = (\frac{1}{a(rs)})v_s\sin\theta\left(n'(rs) + \frac{r}{2}n''(rs)\right) \tag{34}$$

$$K_{r\varphi} = (\frac{1}{a(rs)})\frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\varphi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial X^r}{\partial\varphi}\right] = 0 \tag{35}$$

$$K_{\theta\varphi} = (\frac{1}{a(rs)})\frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{X^\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial X^\theta}{\partial\varphi}\right] = 0 \tag{36}$$

[9] in this case r=rs

For the case of the extrinsic curvatures without a lapse function the dominant terms are the bubble speed vs and the derivatives of the Natario shape function $n'(rs)$ and $n''(rs)$.The first order derivative of the Natario shape function is zero inside and outside the bubble while having a very low value 10^{-52} in the Natario warped region.This is crucial to low the negative energy density requirements as shown in the previous section.Independently of the system of units being used wether in the SI or MKS or the Geometrized System $G = c = 1$ a speed vs of about 200 times faster than light with an order of magnitude of about 10^{21} or 10^{48} would be completely obliterated by the factor 10^{-52} of the Natario shape function first order derivative.For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

But for the case of the extrinsic curvatures with a lapse function that is 1 in the regions inside and outside the Natario bubble while having a large value in the Natario warped region we have an extra factor.The lapse function have a value of 10^{60} in the Natario warped region and $\frac{1}{a(rs)}$ have a value of 10^{-60} contributing also to obliterate the factor 10^{21} or 10^{48}.

The equatios of the negative energy density in the Natario warp drive spacetime given in function of the extrinsic curvatures are the following ones:see pg 3 and 5 in [2].

see also eq 4.33 pg 14 in [34],eqs 21.162(a) to 21.162(c) pg [552(b)] [573(a)] in [11],eqs 2.88 to 2.90 pg [40(b)] [60(a)] in [36],eq 10.2.30 pg [259(b)] [266(a)] in [23],eqs 2.4.5 to 2.4.6 pg [72(b)] [87(a)] in [12].

$$\rho = T_{ab}n^a n^b = \frac{1}{16\pi}\left({}^{(3)}R + (K^i{}_i)^2 - K_{ij}K^{ij}\right) = \frac{1}{16\pi}\left(\theta^2 - K_{ij}K^{ij}\right) \tag{37}$$

Since the trace of the extrinsic curvatures or the expansion of the normal volume elements is zero $\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0$ then $(K^i{}_i)^2 = \theta^2 = 0$ and ${}^{(3)}R = 0$ the equation reduces to:

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{38}$$

The equation above is the negative energy density without lapse function or for a lapse function that is 1 everywhere.For a lapse function that is 1 inside and outside the Natario warp bubble but possessing very large values in the Natario warped region the corresponding equation would then be:(see Appendix I for details)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\frac{1}{a(rs)^2}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{39}$$

The equations are being given in the Geometrized System of units $G = c = 1$ but wether in this system or the SI or MKS System of units the lapse function helps to reduce the negative energy density requirements in the Natario warp drive spacetime.see pg 20 in [34].Note that the factor $\frac{1}{a(rs)^2} = \frac{1}{10^{120}} = 10^{-120}$ thereby reducing effectively the factor 10^{21} or 10^{48} from a speed of 200 times faster than light.

4 The equation of the Natario warp drive spacetime metric with a constant speed vs in the original $3+1$ ADM formalism without a lapse function or using a lapse function always equal to 1 everywhere.

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{40}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{41}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2])(see also Appendix A for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{42}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{43}$$

The covariant shift vector components X_{rs} and X_{θ} are given by:

$$X_{rs} = 2v_s n(rs)\cos\theta \tag{44}$$

$$X_{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{45}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{46}$$

The equation above was written using both contravariant and covariant shift vector components of the Natario vector at the same time.

Since $X_{rs} = X^{rs}$ the equation in the $1 + 1$ spacetime can be written as given below:

- 1)-contravariant form;all the shift vector components of the Natario vector are contravariant

$$ds^2 = (1 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{47}$$

- 2)-covariant form:all the shift vector components of the Natario vector are covariant

$$ds^2 = (1 - (X_{rs})^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{48}$$

The equal contravariant and covariant shift vector component X_{rs} and X^{rs} are then:

$$X^{rs} = X_{rs} = 2v_s n(rs) \tag{49}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.

5 The equation of the Natario warp drive spacetime metric with a constant speed vs in the original $3+1$ ADM formalism using a lapse function α always equal to 1 in the regions inside and outside the Natario bubble but with large values in the Natario warped region.

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix J for details)

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{50}$$

This section is almost similar to the previous section with the Natario vector and the Natario shape function being equal to their counterparts of the previous section however with a difference:the lapse function α

In a $1+1$ spacetime the equatorial plane we get:

$$ds^2 = (\alpha^2 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{51}$$

The equation above was written using both contravariant and covariant shift vector components of the Natario vector at the same time.

Since $X_{rs} = X^{rs}$ the equation in the $1+1$ spacetime can be written as given below:

- 1)-contravariant form;all the shift vector components of the Natario vector are contravariant

$$ds^2 = (\alpha^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{52}$$

- 2)-covariant form:all the shift vector components of the Natario vector are covariant

$$ds^2 = (\alpha^2 - (X_{rs})^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{53}$$

The equal contravariant and covariant shift vector component X_{rs} and X^{rs} are then:

$$X^{rs} = X_{rs} = 2v_s n(rs) \tag{54}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.

6 Differences and resemblances between all these equations with or without lapse function in the original $3+1$ ADM formalism for the Natario warp drive spacetime

The equation in $3+1$ original ADM formalism for the Natario warp drive spacetime for a constant velocity vs without a lapse function or with a lapse function that is always 1 everywhere is given by:(see Appendices A and E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{55}$$

The equation in $3+1$ original ADM formalism for the Natario warp drive spacetime for a constant velocity vs with a lapse function that is always 1 inside and outside the Natario bubble but with a large value in the Natario warped region is given by:(see Appendix J for details)

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{56}$$

Placing both equations side by side one can easily see that the relevant difference between these equations is precisely the lapse function α or in this case α^2.

While the Natario vector or the Natario shape function is the same for both equations the presence of a non unit lapse function α or in this case α^2 affects the spacetime geometry of the second case.

Inside the Natario bubble where the spaceship resides (flat spacetime) the first equation have 1 in the place of the lapse function and the second equation have a lapse function equal to 1 so here we have no differences after all.

Outside the Natario bubble where an external observer resides watching the bubble passing by (also flat spacetime) the first equation again have 1 in the place of the lapse function and the second equation have a lapse function again equal to 1 so here we have no differences after all.

But in the Natario warped region where the derivatives of the Natario shape functions are non-null (curved spacetime) the presence of a large lapse function affects the spacetime geometry and hence the negative energy density of the second case.

The negative energy density for the Natario warp drive in the original $3+1$ ADM formalism without lapse function or with a lapse function always equal to 1 for fixed velocities in the International System of Units SI (see Appendix G) is given by(see pg 5 in [2])

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{57}$$

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$) the negative energy density for fixed velocities is given by:(see Appendix D)

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{58}$$

The negative energy density for the Natario warp drive in the original $3 + 1$ ADM formalism with a large lapse function for fixed velocities in the International System of Units SI (see Appendices G,I and Section 3) is given by(see pg 5 in [2])

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{a(rs)^2}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{59}$$

In the equatorial plane($1 + 1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$) the negative energy density for fixed velocities is given by:(see Appendices D and I)

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{a(rs)^2}\left[3(n'(rs))^2\right] \tag{60}$$

Wether in the 3+1 or 1+1 spacetimes the first equation do not possesses the factor $\frac{1}{a(rs)^2}$ but the second equation possesses this factor.A large lapse function reduces the negative energy density requirements in the second case.see pg 20 in [34]

Note that in both equations the negative energy density requirements even in the equatorial plane is not null The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg [116(a)][116(b)] in [26])[10].

But for the case of the warp drive equations with variable velocities vs (see [20],[30] and [33]) wether the lapse function $a(rs)$ is 1 or not we can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.Or we can use the techniques described in the Appendix I using extrinsic curvatures and shift vectors but still subject to computation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pgs [342(b)] or [369(a)] in [11], pgs [276(b)] or [294(a)] in [13],pgs [454, 457, 560(b)] or [465, 468, 567(a)] in [14] pg [98(a)] or [98(b)] in [25],pgs [183(a)] or [178(b)] in [27]).

Appendix C pgs [551 − 555(b)] or [559 − 563(a)] in [14] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.

Also Barak Shoshany and Ben Snodgrass used *OGRe* for $Mathemathica$ and *OGRePy* for *Python* to compute relevant tensor calculations.see pg 5 in [34].

[10] see Appendices B,C,F and H

But since the Natario shape function $n(rs)$ is the same for all these equations it is reasonable to suppose that derivatives of first second(or perhaps higher)order will appear in the negative energy density expression for the Natario warp drive with variable velocity and since the derivatives of first or second order for the Natario shape function possesses extremely low values these values can obliterate large terms for velocities vs or large accelerations a.For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

7 Conclusion:

In this work borrowed the ideas of Barak Shoshany and Ben Snodgrass in [34] and we presented a continuous and differentiable lapse function α for the Natario warp drive with zero expansion in [2].

Our expression for α is given by:

$$a(rs) = (\frac{1}{2}[1 - (tanh[@(rs - R)])^{(2)}])^{(-WF)} \tag{61}$$

$$a(rs) = (\frac{1}{2}[1 - (tanh[@(rs - R)])^{(2)}])^{(\frac{1}{WF})} \tag{62}$$

This expression for the lapse function α was constructed using the Alcubierre and Natario shape functions as inspirations and possesses a very interesting behavior:

- inside the Natario warp bubble where the spaceship resides (flat spacetime) the lapse function is always 1.
- outside the Natario warp bubble where an external observer resides watching the bubble passing by (also flat spacetime) the lapse function is again always 1.
- in the Natario warped region where the derivatives of the Natario shape function are not null (curved spacetime) the lapse function possesses very large values.

Our choice for the Natario shape function is given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{63}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{64}$$

The shape function above gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]) and we used the Alcubierre shape function to define the Natario shape function.

Considering a Natario warp drive moving with $vs = 200$ which means to say 200 times faster than light in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) and since the negative energy density ia proportional to the factor $\frac{c^2}{G}\frac{v_s^2}{8\pi}$ (as we have shown in Appendix G) we would get in the expression of the negative energy density the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[11] of about $6 \times 10^{24} kg$

[11] see Wikipedia:The free Encyclopedia

This term 10^{48} is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the square derivative of our choosed Natario shape function possesses values of 10^{-102} (as we have shown in section 2) ameliorating the factor 10^{48} making the warp drive negative energy density more "affordable". For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

The equation in $3+1$ original ADM formalism for the Natario warp drive spacetime for a constant velocity vs with a lapse function that is always 1 inside and outside the Natario bubble but with a large value in the Natario warped region is given by:(see Appendix J for details)

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{65}$$

The negative energy density for the Natario warp drive in the original $3+1$ ADM formalism with a large lapse function for fixed velocities in the International System of Units SI (see Appendices G,I and Section 3) is given by(see pg 5 in [2])

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{a(rs)^2}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2 \sin^2\theta\right] \tag{66}$$

Our lapse function that is 1 in the regions inside and outside the Natario bubble while having a large value in the Natario warped region possesses an an extra factor.The lapse function have a value of 10^{60} in the Natario warped region and $\frac{1}{a(rs)}$ have a value of 10^{-60} contributing also to obliterate the factor 10^{48}. Note that the negative energy density incorporates the factor $\frac{1}{a(rs)^2} = \frac{1}{10^{120}} = 10^{-120}$ thereby reducing effectively the factor 10^{48} from a speed of 200 times faster than light(as we have shown in section 3) .

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$) the negative energy density is given by:(see Appendices D and I)

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{a(rs)^2}\left[3(n'(rs))^2\right] \tag{67}$$

Note that the negative energy density requirements even in the equatorial plane is not null The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg [116(a)][116(b)] in [26])[12].

[12]see Appendices B,C,F and H

A real and fully functional warp drive must encompasses accelerations or de-accelerations in order to go from 0 to 200 times light speed in the beginning of an interstellar journey and to slow down to 0 again in the end of the interstellar journey. But for the case of the warp drive equations with variable velocities vs (see [20],[30] and [33]) we still do not have geometries possessing a lapse function α.This(perhaps) will be done in future works.

Because collisions between the walls of the warp bubble and the hazardous particles of the Interstellar Medium(IM) would certainly occurs in a real superluminal interstellar spaceflight we borrowed the more than welcome idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection of the spaceship and the crew members in the Natario warp drive equation for constant speed vs(see pg 46 in [18],pg 3 in [19]).

Our idea was to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the dangerous IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

A submicroscopic outer radius of the bubble being the only part in contact with our Universe would mean a submicroscopic surface exposed to the collisions against the hazardous IM particles thereby reducing the probabilities of dangerous impacts against large objects (comets asteroids etc) enhancing the protection level of the spaceship and hence the survivability of the crew members.

Any future development for the Natario warp drive must encompass the more than welcome idea of Chris Van Den Broeck and this idea can also be easily implemented in the Natario warp drive with a lapse function .Since the Broeck idea is independent of the Natario geometry possessing a lapse function we did not covered the Broeck idea here because it was already covered in [18] and [19] and in order to discuss the geometry of a Natario warp drive with a lapse function the Broeck idea is not needed here however the Broeck idea must appear in a real Natario warp drive a lapse function concerning realistic superluminal interstellar spaceflights.see also [32]

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we do not know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.And we also so not know how to generate the lapse function either.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario-Broeck warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario-Broeck warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

8 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [2],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [15],eq 3.72 pg 69$(a)(b)$ in [15]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{68}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{69}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{70}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{71}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{72}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{73}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see pg 8 in [4],eq 3.72 pg 69$(a)(b)$ in [15]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{74}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{75}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{76}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{77}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{78}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{79}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{80}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{81}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{82}$$

We know that the following expression holds true(see pg 9 in [3], eq 3.79 pg 70$(a)(b)$ in [15]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{83}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{84}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{85}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{86}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{87}$$

According to pg 10 in [3],eq 3.90 pg 74$(a)(b)$ in [15] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{88}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{89}$$

Because and according to pg 10 in [3],eqs 3.90 and 3.91 pg 74(a)(b) in [15],tb 3.2 pg 68(a)(b) in [15]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{90}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{91}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{92}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{93}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{94}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{95}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{96}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{97}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{98}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{99}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \quad (100)$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(d\theta\wedge d\varphi) \quad (101)$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(d\varphi\wedge dr)\sim -r\sin\theta(dr\wedge d\varphi) \quad (102)$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dr\wedge(rd\theta)\sim r(dr\wedge d\theta) \quad (103)$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \quad (104)$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \quad (105)$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \quad (106)$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (107)$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (108)$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \quad (109)$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \quad (110)$$

With our pedagogical approaches

$$nX=2vs(t)f(r)\ cos\theta dr-vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \quad (111)$$

$$nX=-2vs(t)f(r)\ cos\theta dr+vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \quad (112)$$

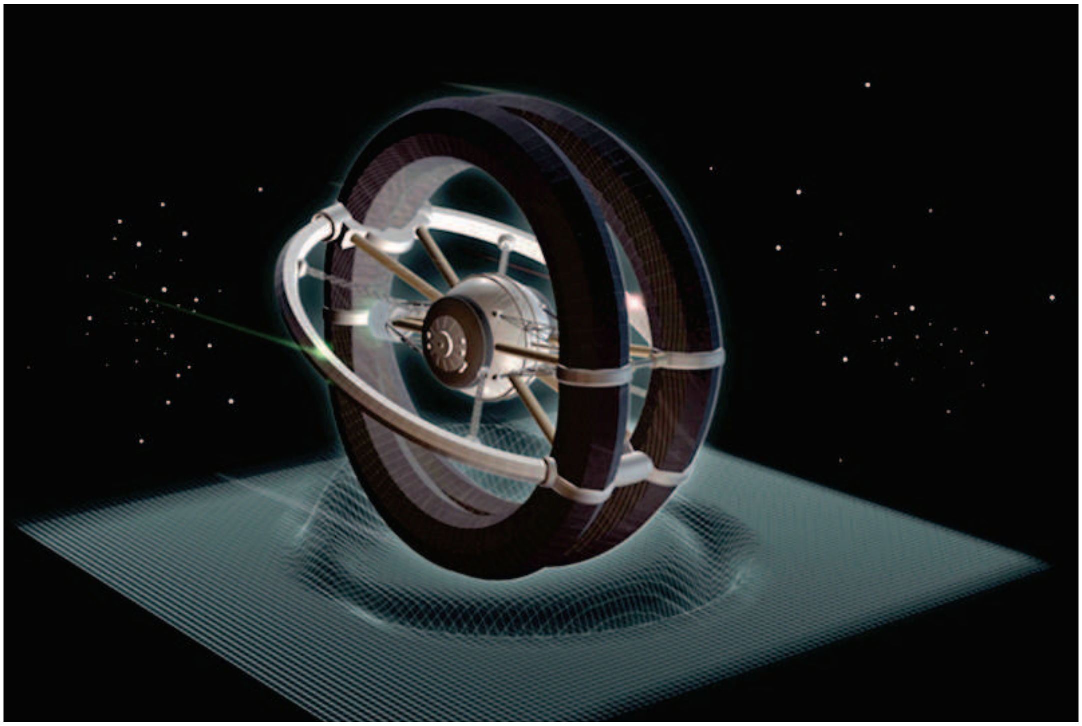

Figure 1: Artistic representation of the Natario warp drive .Note in the bottom of the figure the Alcubierre expansion of the normal volume elements .(Source:Internet)

9 Appendix B:Artistic Presentation of the Natario warp drive

According to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).(pg 5 in [2]).

The expansion of the normal volume elements in the Natario warp drive is given by the following expressions(pg 5 in [2]).

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{113}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{114}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{115}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{116}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.

This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The "bars" in the figure were included to illustrate how the expansion in one direction can be counterbalanced by the contraction in the other directions.These "bars" keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [1] is also included

Note also that the energy density in the Natario warp drive $3+1$ spacetime being given by the following expressions(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2 \cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2 \sin^2\theta\right]. \tag{117}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2 \cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2 \sin^2\theta\right]. \tag{118}$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons or other particles so these will not reach the ship inside the bubble.The illustrated "bars" are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble.[13]

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg [116(a)][116(b)] in [26])

-)-Energy directly above the ship($y - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[\left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2 \sin^2\theta\right]. \tag{119}$$

-)-Energy directly in front of the ship($x - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2 \cos^2\theta\right]. \tag{120}$$

[13] See also Appendix C

Note also that even in a $1+1$ dimensional spacetime the Natario warp drive retains the zero expansion behavior:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{121}$$

$$K_{\theta\theta} = \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{122}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} = v_s n'(r)\cos\theta \tag{123}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{124}$$

In all these equations the term r is our term rs

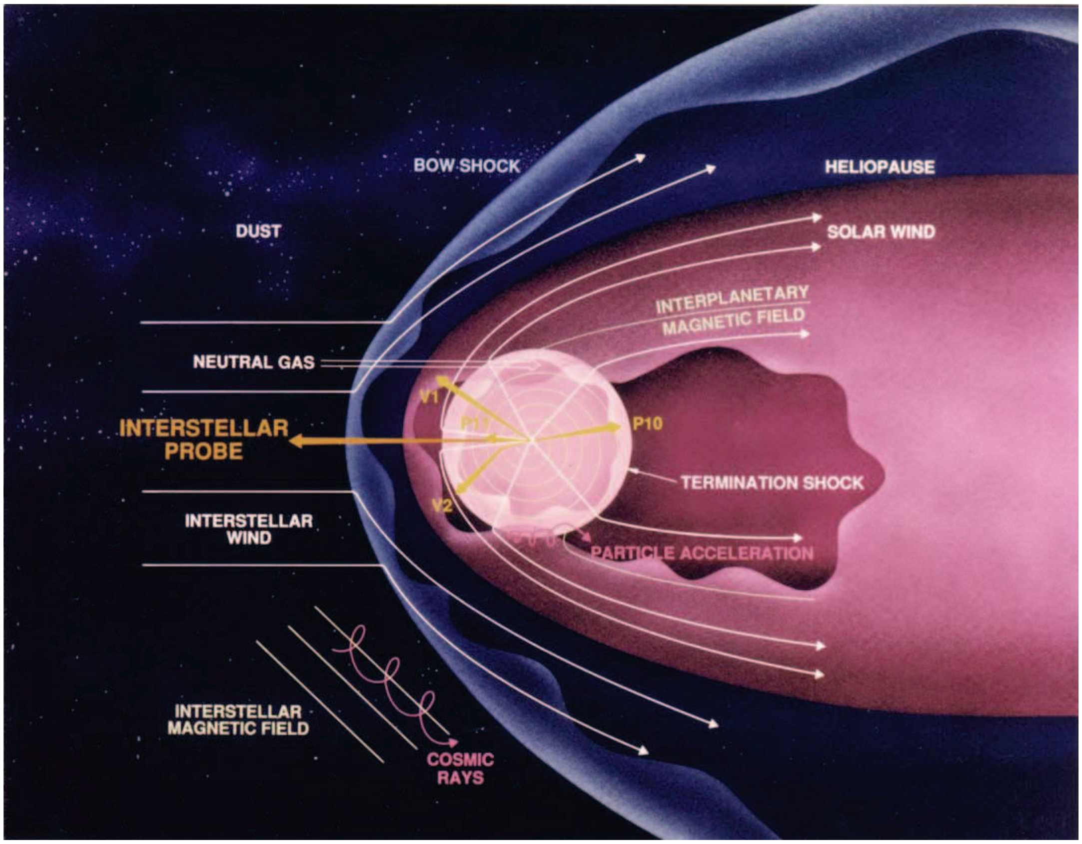

Figure 2: Artistic representation of a Natario warp wrive in a real superluminal space travel .Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter(brown arrows).(Source:Internet)

10 Appendix C:Artistic Presentation of a Natario warp drive in a real faster than light interstellar spaceflight

Above is being presented the artistic presentation of a Natario warp drive in a real interstellar superluminal travel.The "ball" or the spherical shape is the Natario warp bubble with the negative energy surrounding the ship in all directions and mainly protecting the front of the bubble.[14]

The brown arrows in the front of the Natario bubble are a graphical presentation of the negative energy in front of the ship deflecting interstellar dust,neutral gases,hydrogen atoms,interstellar wind photons etc.[15]

The spaceship is at the rest and in complete safety inside the Natario bubble.

[14] See Appendix B

[15] see Appendices F and H for the composition of the Interstellar Medium IM)

In order to allow to the negative energy density of the Natario warp drive the deflection of incoming hazardous particles from the Interstellar Medium(IM) the Natario warp drive energy density must be heavier or denser when compared to the IM density.

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg [116(a)][116(b)] in [26])

11 Appendix D:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[16]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{125}$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{126}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{127}$$

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg [116(a)][116(b)] in [26])

[16] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

12 Appendix E:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg [64(b)] [79(a)] in [12])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65(b)] [80(a)] in [12].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12])
(see also fig 21.2 pg [506(b)] [533(a)] in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg [507(b)] [534(a)] in [11])[17]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[17]we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11] with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{128}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{129}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{130}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{131}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{132}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{133}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{134}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{135}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{136}$$

$$(dx^i)^2 = dx^i dx^i \tag{137}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{138}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{139}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{140}$$

Note that the expression above is exactly the eq (2.2.4) pgs $[67(b)]$ $[82(a)]$ in [12].It also appears as eq 1 pg 3 in [1].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{141}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{142}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{143}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{144}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{145}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{146}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{147}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{148}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{149}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{150}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{151}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{152}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{153}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{154}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{155}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{156}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{157}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{158}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{159}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{160}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{161}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{162}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{163}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{164}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{165}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs)\cos\theta = X^r = X^{rs} \tag{166}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{167}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i) dt^2 + 2 X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{168}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{169}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{170}$$

The matrix components 2×2 evaluated separately for rs and θ gives the following results:[18]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0r} \\ g_{r0} & g_{rr} \end{pmatrix} = \begin{pmatrix} 1 - X_r X^r & X_r \\ X_r & -\gamma_{rr} \end{pmatrix} \tag{171}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0r} \\ g^{r0} & g^{rr} \end{pmatrix} = \begin{pmatrix} 1 & X^r \\ X^r & -\gamma^{rr} + X^r X^r \end{pmatrix} \tag{172}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0\theta} \\ g_{\theta 0} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 - X_\theta X^\theta & X_\theta \\ X_\theta & -\gamma_{\theta\theta} \end{pmatrix} \tag{173}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0\theta} \\ g^{\theta 0} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & X^\theta \\ X^\theta & -\gamma^{\theta\theta} + X^\theta X^\theta \end{pmatrix} \tag{174}$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i) dt^2 + 2 X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{175}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta) dt^2 + 2 (X_{rs} drs dt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{176}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta) dt^2 + 2 (X_{rs} drs + X_\theta d\theta) dt - drs^2 - rs^2 d\theta^2 \tag{177}$$

[18] Actually we know that the real matrix is a 3×3 matrix with dimensions t rs and θ.Our 2×2 approach is a simplification

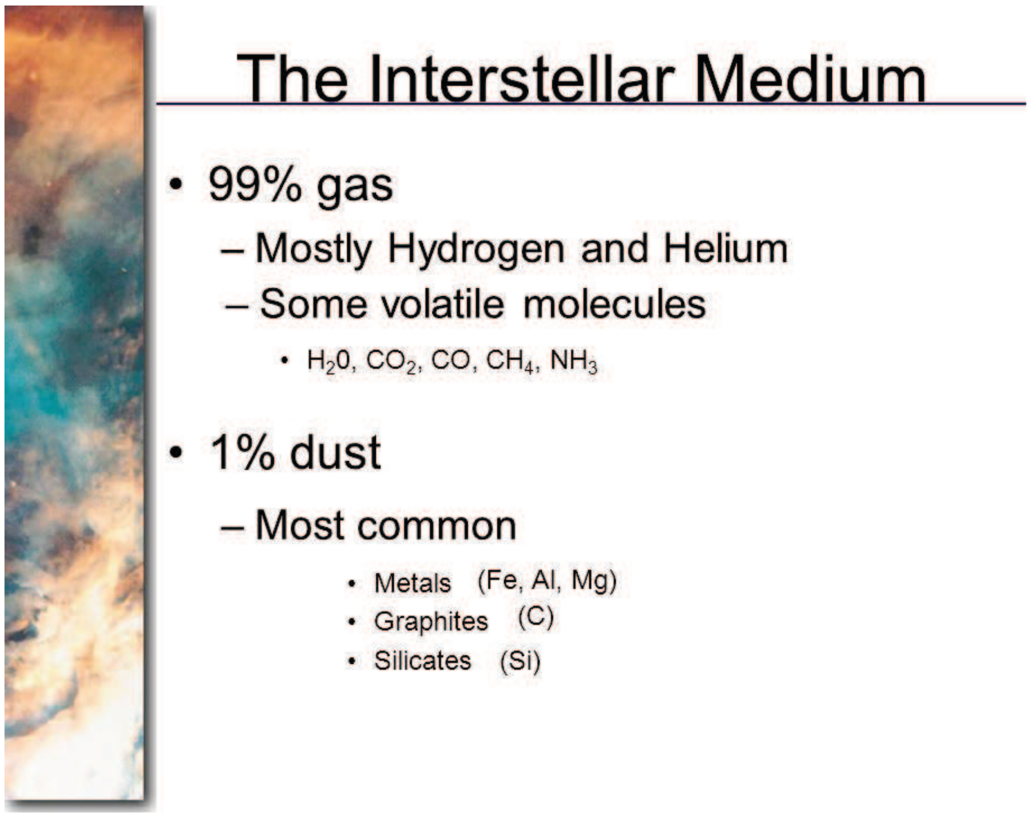

Figure 3: Composition of the Interstellar Medium IM(Source:Internet)

13 Appendix F:Composition of the Interstellar Medium IM

14 Appendix G:Dimensional Reduction from $\frac{c^4}{G}$ to $\frac{c^2}{G}$

The Alcubierre expressions for the Negative Energy Density in Geometrized Units $c = G = 1$ are given by(pg 4 in [2])(pg 8 in [1]):[19]:

$$\rho = -\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{178}$$

$$\rho = -\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{179}$$

In this system all physical quantities are identified with geometrical entities such as lengths,areas or dimensionless factors.Even time is interpreted as the distance travelled by a pulse of light during that time interval,so even time is given in lengths.Energy,Momentum and Mass also have the dimensions of lengths.We can multiply a mass in kilograms by the conversion factor $\frac{G}{c^2}$ to obtain the mass equivalent in meters.On the other hand we can multiply meters by $\frac{c^2}{G}$ to obtain kilograms.The Energy Density($\frac{Joules}{meters^3}$)in Geometrized Units have a dimension of $\frac{1}{length^2}$ and the conversion factor for Energy Density is $\frac{G}{c^4}$.Again on the other hand by multiplying $\frac{1}{length^2}$ by $\frac{c^4}{G}$ we retrieve again ($\frac{Joules}{meters^3}$). [20].

This is the reason why in Geometrized Units the Einstein Tensor have the same dimension of the Stress Energy Momentum Tensor(in this case the Negative Energy Density)and since the Einstein Tensor is associated to the Curvature of Spacetime both have the dimension of $\frac{1}{length^2}$.

$$G_{00} = 8\pi T_{00} \tag{180}$$

Passing to normal units and computing the Negative Energy Density we multiply the Einstein Tensor (dimension $\frac{1}{length^2}$) by the conversion factor $\frac{c^4}{G}$ in order to retrieve the normal unit for the Negative Energy Density ($\frac{Joules}{meters^3}$).

$$T_{00} = \frac{c^4}{8\pi G} G_{00} \tag{181}$$

Examine now the Alcubierre equations:

$vs = \frac{dxs}{dt}$ is dimensionless since time is also in lengths.$\frac{y^2+z^2}{rs^2}$ is dimensionless since both are given also in lengths. $f(rs)$ is dimensionless but its derivative $\frac{df(rs)}{drs}$ is not because rs is in meters. So the dimensional factor in Geometrized Units for the Alcubierre Energy Density comes from the square of the derivative and is also $\frac{1}{length^2}$.Remember that the speed of the Warp Bubble vs is dimensionless in Geometrized Units and when we multiply directly $\frac{1}{length^2}$ from the Negative Energy Density in Geometrized Units by $\frac{c^4}{G}$ to obtain the Negative Energy Density in normal units $\frac{Joules}{meters^3}$ the first attempt would be to make the following:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{182}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{183}$$

[19]See Geometrized Units in Wikipedia

[20]See Conversion Factors for Geometrized Units in Wikipedia

But note that in normal units vs is not dimensionless and the equations above do not lead to the correct dimensionality of the Negative Energy Density because the equations above in normal units are being affected by the dimensionality of vs.

In order to make vs dimensionless again,the Negative Energy Density is written as follows:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{184}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{185}$$

Giving:

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{186}$$

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{187}$$

As already seen.The same results are valid for the Natario Energy Density

Note that from

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{188}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{189}$$

Making $c = G = 1$ we retrieve again

$$\rho = -\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{190}$$

$$\rho = -\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{191}$$

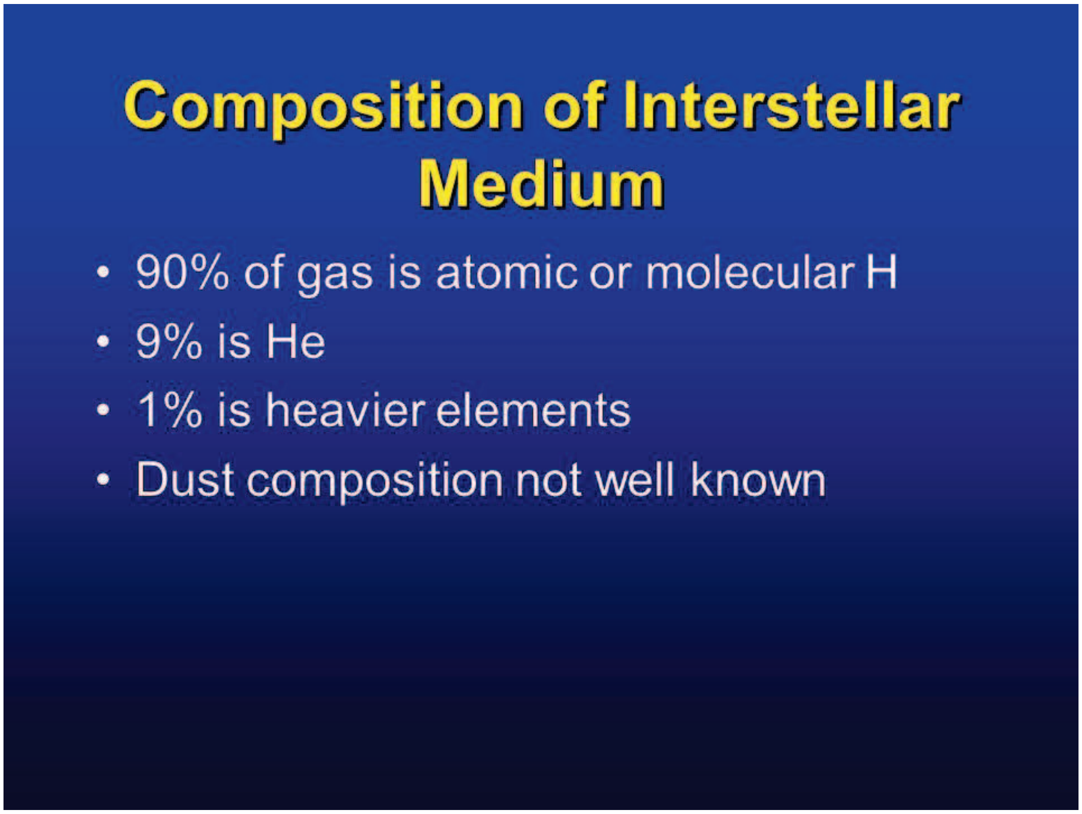

Figure 4: Composition of the Interstellar Medium IM(Source:Internet)

15 Appendix H:Composition of the Interstellar Medium IM

16 Appendix I:A generic procedure to compute extrinsic curvatures and energy densities with lapse functions using mainly the Shoshany-Snodgrass work

We will start with the $4D$ ADM metric where N stands for the lapse function β^i and β^j are shift vectors and δ_{ij} or γ_{ij} are the induced and diagonalized $3D$ ADM metrics in agreement with the Appendix E.see eq 5.1 pg 15,eq 8.1 pg 33 in [34].
see also eq 21.40 pg [507(b)] [528(a)] in [11],eq 2.123 pg [45(b)] [65(a)] in [36]

$$ds^2 = -N^2 dt^2 + \delta_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt) \tag{192}$$

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt) \tag{193}$$

The $4D$ ADM matrices are given by:see eqs $A.1$ and $A.2$ pg 34 in [34]
see also eq 21.42 pg [507(b)] [528(a)] and 21.44 pg [508(b)] [529(a)] in [11],eqs 2.119 and 2.122 pg [45(b)] [65(a)] in [36],eqs 2.2.5 and 2.2.6 pg [67(b)] [82(a)] in [12]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + \beta_k\beta^k & -\beta_j \\ -\beta_i & \gamma_{ij} \end{pmatrix} \tag{194}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & -\frac{\beta^j}{N^2} \\ -\frac{\beta^i}{N^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{N^2} \end{pmatrix} \tag{195}$$

The components of the normal vector are given by:see eq $A.3$ pg 34 in [34]

$$n_\mu = (-N, 0), n^\mu = \frac{1}{N}(1, \beta), \tag{196}$$

In the contravariant matrix above $g^{\mu\nu}$ γ^{ij} is the inverse of the $3D$ induced metric and $\beta_i \equiv \gamma_{ij}\beta^j$.

The $3D$ induced and diagonalized metrics are $g_{ij} = \gamma_{ij} = \delta_{ij}$ and the shift vectors β^i and β^j refers to the $3D$ scripts $i, j = 1, 2, 3$.The $4D$ ADM scripts are $\mu, \nu = 0, 1, 2, 3$.

Now we replace $\mu \to i$, $\nu \to j$, in the components of the normal vector and use $n_i = 0$ since $i, j = 1, 2, 3$ and $n_\mu = (-N, 0)$ with 0 corresponds to the scripts $i, j = 1, 2, 3$ to get the equation of the extrinsic curvature K_{ij} for the $3D$ scripts $i, j = 1, 2, 3$ as a covariant derivative of $n_i = 0$.

see eq $B.12$ pg 36 in [34]

$$K_{ij} = \nabla_{(i}n_{j)}. \tag{197}$$

see eq $B.13$ pg 36 in [34]

$$K_{ij} = \partial_{(i}n_{j)} - \Gamma^\rho_{(ij)}n_\rho \tag{198}$$

Since $i, j = 1, 2, 3$ the scripts of the $3D$ induced metric and $n_i = n_j = 0$ the ordinary symmetrical derivative $\partial_{(i}n_{j)}$ vanishes and the Christoffel symbol becomes the dominant term.Then the component of the normal vector $n_\mu = (-N, 0)$ that accounts is the term $n_0 = (-N)$ with the $4D$ script $\mu = 0$ and the contravariant script of the Christoffel symbol must only be 0.

The equation now becomes:

$$K_{ij} = 0 + N\Gamma^0_{ij} \tag{199}$$

The expansion of the Christoffel symbol gives:

$$K_{ij} = \frac{1}{2}N\left(g^{00}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}) + g^{0k}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})\right) \tag{200}$$

Inserting the components of the $4D$ ADM matrices
$g_{0j} = -\beta_j$,$g_{0i} = -\beta_i$,$g^{00} = -\frac{1}{N^2}$,$g^{0k} = (-\frac{1}{N^2}\beta^k)$ in the expression above we get the following expression:

$$K_{ij} = \frac{1}{2}N\left(-\frac{1}{N^2}(\partial_i(-\beta_j) + \partial_j(-\beta_i) - \partial_0\delta_{ij}) + (-\frac{1}{N^2}\beta^k)(\partial_i\delta_{kj} + \partial_j\delta_{ki} - \partial_k\delta_{ij})\right) \tag{201}$$

The equation reduces to:see eqs $B.13$ pg 36 and 4.28 pg 13 in [34]
see also eqs 9 and 10 pg 5 in [1],eq 21.67 pg [513(b)],[534(a)] in [11],eq 2.3.12 pg [71(b)],[86(a)] in [12],eq 2.128 pg [46(b)],[66(a)] in [36]

$$K_{ij} = \frac{1}{2}\frac{1}{N}\left((\partial_i\beta_j + \partial_j\beta_i) + 0\right) = \frac{1}{N}\partial_{(i}\beta_{j)}, \tag{202}$$

$$K_{ij} = \frac{1}{2}\frac{1}{N}\left(\partial_i\beta_j + \partial_j\beta_i\right) = \frac{1}{N}\partial_{(i}\beta_{j)}, \tag{203}$$

Note that for a lapse function $N = 1$ the equation of the extrinsic curvature reduces to:

$$K_{ij} = \frac{1}{2}\left(\partial_i\beta_j + \partial_j\beta_i\right) = \partial_{(i}\beta_{j)}, \tag{204}$$

The trace of the extrinsic curvature is given by $K = K^\rho_\rho = K^i_i$.see pg 14 in [34] and the factor $\frac{1}{N}$ appears in the trace.Considering the square of the trace the corresponding factor would then be $\frac{1}{N^2}$.Also the term $K_{ij}K^{ij}$ have a factor $\frac{1}{N^2}$ too.

The energy density in function of the extrinsic curvatures is given by: see eq 4.33 pg 14 in [34] [21]

$$\rho \equiv \frac{1}{8\pi G}G_{nn} = \frac{1}{16\pi G}(K^2 - K_{ij}K^{ij}) \tag{205}$$

Compare with the equation of the energy density given in pg 3 in [2].[22]
see also eq 4.33 pg 14 in [34],eqs 21.162(a) to 21.162(c) pg [552(b)] [573(a)] in [11],eqs 2.88 to 2.90 pg [40(b)] [60(a)] in [36],eq 10.2.30 pg [259(b)] [266(a)] in [23],eqs 2.4.5 to 2.4.6 pg [72(b)] [87(a)] in [12].

$$\rho = T_{ab}n^a n^b = \frac{1}{16\pi}\left({}^{(3)}R + (K^i{}_i)^2 - K_{ij}K^{ij}\right) = \frac{1}{16\pi}\left(\theta^2 - K_{ij}K^{ij}\right) \tag{206}$$

With the trace of the extrinsic curvatures or the expansion of the normal volume elements being $\theta = K = K^\rho_\rho = K^i_i$ then $(K^i{}_i)^2 = \theta^2 = K^2$ and ${}^{(3)}R = 0$

[21] given in a system of units where $G \neq 1$
[22] given in the Geometrized system of units $G = c = 1$

Generic equations for the energy density where the factor $\frac{1}{N^2}$ and ordinary derivatives of the shift vectors appears are given by:see eqs 5.3 and 5.4 pg 15 in [34].[23]

$$\rho = \frac{1}{16\pi G}\frac{1}{N^2}((\partial_i\beta_i)^2 - \partial_i\beta_j\partial_i\beta_j), \tag{207}$$

$$\rho = \frac{1}{16\pi G}\frac{1}{N^2}\left(\partial_i(\beta_i\partial_j\beta_j - \beta_j\partial_j\beta_i) - \partial_{[i}\beta_{j]}\partial_{[i}\beta_{j]}\right), \tag{208}$$

[23]given in a system of units where $G \neq 1$

17 Appendix J:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre using a lapse function α

This Appendix is a continuation of the Appendix E except for the fact that we do not suppress the lapse function here.Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11] with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-, +, +, +)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{209}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{210}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{211}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{212}$$

But since $dl^2 = \gamma_{ij}dx^i dx^j$ is the ADM induced metric and must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{213}$$

Expanding the square term and recombining all the terms we have:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{214}$$

Note that the expression above is exactly the eq (2.2.4) pgs $[67(b)]$ $[82(a)]$ in [12].It also appears as eq 1 pg 3 in [1].

Changing the signature from $(-, +, +, +)$ to signature $(+, -, -, -)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{215}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{216}$$

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{217}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{218}$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original $3+1$ ADM formalism with a lapse function is given by:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{219}$$

Inserting the components of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{220}$$

We have:

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drsdt + X_{\theta}d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{221}$$

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{222}$$

The lapse function is equal to 1 inside and outside the Natario warp bubble while having large values in the Natario warped region.

18 Remarks

References [11],[12],[13],[14],[15],[22],[23],[24],[25],[26],[27],[28] and [36] are standard textbooks used to study General Relativity or warp drive spacetimes and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [11] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader.

All the numerical plots presented in this work are available for those interested to check out our results.We gladly provide the files.These plots also present the results mentioned in [18].

We used the first version of Barak Shoshany and Ben Snodgrass work arXiv:2309.10072v1 (gr-qc) 18 Sep 2023 to assemble our own work.We will adapt our work if future versions of their work appears.

19 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[24]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[25][26]

[24]special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[25]"Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[26]appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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