Cool formula for Pi

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ABSTRACT: In this note we consider the sum: $1-\frac{1}{9}+\frac{1}{45}-\frac{1}{189}+\ldots$

Keywords: Leibniz-Gregory-Madhava series, Sharp’s series, number Pi.

1. Introduction: The number Pi

Eq.(1):

$$\pi=\lim_{n\longrightarrow\infty} 2^{n+1}\sin\left(\frac{\pi}{2^{n+1}}\right)=3.1415926535\ldots$$

2. Leibniz-Gregory-Madhava series

Eq.(2):

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Remark 1:

$$n\in\{0,1,2,3,4,\ \ldots\}\Longrightarrow 2n+1\in\{1,3,5,7,9,\ \ldots\}$$

3. Sharp’s series

Eq.(3):

$$\frac{\pi}{2\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)}=1-\frac{1}{9}+\frac{1}{45}-\frac{1}{189}\ldots$$

Remark 2:

$$n\in\{0,1,2,3,4,\ \ldots\}\Longrightarrow 3^n(2n+1)\ \in\{1,3,5,7,9,\ \ldots\}$$

Remark 3:

$$3^n(2n+1)=2\cdot 3^n n+3^n-1+1=2\left(3^n n+\frac{3^n-1}{2}\right)+1=2m+1\,,\ m=3^n n+\frac{3^n-1}{2}$$

$$n\in\{0,1,2,3,4,\ \ldots\}\Longrightarrow m=3^n n+\frac{3^n-1}{2}\in\{0,1,2,3,4,\ \ldots\}$$

Remark 4:

$$\frac{\pi}{2\sqrt{3}} = \sum_{m=0}^{\infty} \frac{c(m)}{2m+1} \;,\; c(m) = \begin{cases} (-1)^n & m = n\,3^n + \frac{3^n-1}{2} \;,\; n \in \{0, 1, 2, 3, \ldots\} \\ 0 & m \neq n\,3^n + \frac{3^n-1}{2} \;,\; n \in \{0, 1, 2, 3, \ldots\} \end{cases}$$

Remark 5:

$$\frac{\pi}{2\sqrt{3}} + \frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{c(m) + (-1)^m}{2m+1}$$

Remark 6:

$$\frac{\pi}{2\sqrt{3}} + \frac{\pi}{4} = \sum_{n=0}^{\infty} \left(\frac{2}{3^{2n}(4n+1)} + \sum_{k=m(2n)+1}^{m(2n+1)-1} \frac{(-1)^k}{2k+1} + \sum_{k=m(2n+1)+1}^{m(2n+2)-1} \frac{(-1)^k}{2k+1} \right)$$

$$m(n) = n\,3^n + \frac{3^n-1}{2} \;,\; n \in \{0, 1, 2, 3, \ldots\}$$

4. Cool formula for Pi

Eq.(4):

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{3^{-2}}{1} + \frac{3^{-4}}{1-3^{-1}-3^{-2}} - \frac{3^{-5}}{1-3^{-1}+3^{-2}} + \frac{3^{-6}}{1} - \frac{3^{-7}}{1+3^{-1}-3^{-2}} + \frac{3^{-8}}{1+3^{-1}+3^{-2}} - \frac{3^{-10}}{1-3^{-1}-3^{-2}} +$$
$$\frac{3^{-11}}{1-3^{-1}-3^{-3}} - \frac{3^{-12}}{1-3^{-1}+3^{-3}} + \frac{3^{-13}}{1-3^{-1}+3^{-2}} - \frac{3^{-14}}{1-3^{-2}-3^{-3}} + \frac{3^{-15}}{1-3^{-2}+3^{-3}} - \frac{3^{-16}}{1} +$$
$$\frac{3^{-17}}{1+3^{-2}-3^{-3}} - \ldots + \frac{3^{-24}}{1-3^{-1}-3^{-2}-3^{-3}-3^{-4}} - \ldots + \frac{3^{-34}}{1-3^{-1}+3^{-2}-3^{-3}+3^{-4}} - \ldots$$

Remark 7:

$$a_n = 2n+1 \;,\; n \in \{0, 1, 2, 3, \ldots\}$$
$$a_0 = 1 = 3^0$$
$$a_1 = 3 = 3^1$$
$$a_2 = 5 = 3^2 - 3^1 - 1$$
$$a_3 = 7 = 3^2 - 3^1 + 1$$
$$a_4 = 9 = 3^2$$
$$a_5 = 11 = 3^2 + 3^1 - 1$$
$$a_6 = 13 = 3^2 + 3^1 + 1$$
$$a_7 = 15 = 3^3 - 3^2 - 3^1$$
$$a_8 = 17 = 3^3 - 3^2 - 1$$
$$a_9 = 19 = 3^3 - 3^2 + 1$$
$$a_{10} = 21 = 3^3 - 3^2 + 3^1$$
$$a_{20} = 41 = 3^4 - 3^3 - 3^2 - 3^1 - 1$$
$$a_{30} = 61 = 3^4 - 3^3 + 3^2 - 3^1 + 1$$

$$a_{50} = 101 = 3^4 + 3^3 - 3^2 + 3^1 - 1$$
$$a_{100} = 201 = 3^5 - 3^4 + 3^3 + 3^2 + 3^1$$
$$a_{150} = 301 = 3^5 + 3^4 - 3^3 + 3^1 + 1$$
$$a_{200} = 401 = 3^6 - 3^5 - 3^4 - 3^1 - 1$$
$$a_{350} = 701 = 3^6 - 3^3 - 1$$

Remark 8:

$$a_n = 2\,n + 1 \Longrightarrow a_{n+m} = a_n + a_m - 1\ ,\ n, m \in \{0, 1, 2, 3, \ ...\}$$
$$a_{25} = 51 = a_{20} + a_5 - 1 = \left(3^4 - 3^3 - 3^2 - 3^1 - 1\right) + \left(3^2 + 3^1 - 1\right) - 1 = 3^4 - 3^3 - 3^1$$
$$a_{35} = 71 = a_{30} + a_5 - 1 =$$
$$\left(3^4 - 3^3 + 3^2 - 3^1 + 1\right) + \left(3^2 + 3^1 - 1\right) - 1 = 3^4 - 3^3 + 2 \cdot 3^2 - 1 = 3^4 - 3^3 + (3 - 1) \cdot 3^2 - 1 = 3^4 - 3^2 - 1$$

Remark 9:

$$a_n = 2\,n + 1 \Longrightarrow \begin{cases} a_{3\,n} = 3\,a_n - 3 + 1 & , n \in \{0, 1, 2, 3, \ ...\} \\ a_{3\,n+1} = 3\,a_n & , n \in \{0, 1, 2, 3, \ ...\} \\ a_{3\,n+2} = 3\,a_n + 3 - 1 & , n \in \{0, 1, 2, 3, \ ...\} \end{cases}$$

Remark 10:

$$a_n = 2\,n + 1 \Longrightarrow a_{3^m\,n} = 3^m\,a_n - 3^m + 1\ ,\ \ n, m \in \{0, 1, 2, 3, \ ...\}$$
$$a_n = 2\,n + 1 \Longrightarrow a_{(n+1)^2} = a_{n^2} + 2\,a_n\ ,\ n \in \{0, 1, 2, 3, \ ...\}$$
$$a_n = 2\,n + 1 \Longrightarrow a_{2\,n} = 2\,a_n - 1\ ,\ n \in \{0, 1, 2, 3, \ ...\}$$
$$a_n = 2\,n + 1 \Longrightarrow a_{2\,n+1} = 2\,a_n + 1\ ,\ n \in \{0, 1, 2, 3, \ ...\}$$
$$a_n = 2\,n + 1 \Longrightarrow a_{3^n} = 3^{n+1} - 3^n + 1\ ,\ \ n \in \{0, 1, 2, 3, \ ...\}$$
$$a_n = 2\,n + 1 \Longrightarrow a_n\,a_m = a_{2\,n\,m+n+m} = a_{2\,n\,m} + a_{n+m} - 1\ ,\ \ n, m \in \{0, 1, 2, 3, \ ...\}$$
$$a_n = 2\,n + 1 \Longrightarrow a_{n\,m} = a_n + a_{n\,(m-1)} - 1\ ,\ n, m \in \{1, 2, 3, 4, \ ...\}$$

5. References

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