Source-Free Conformal Waves on Spacetime

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Abstract

Investigating conformal metrics on (Pseudo-)Riemannian spaces in any number of dimensions, it is shown that the pure scalar curvature R as the Lagrange density leads to a homogeneous d'Alembert equation on spacetime which allows for source-free wave phenomena.

This suggests to use the scalar curvature R itself rather than the Hilbert-Einstein action $R\sqrt{|g|}$ as the governing Lagrange density for General Relativity to also find general, non-conformal solutions.

Keywords: General Relativity, Conformal Metric, Action Principle

1 Notation

Tensors are represented in index notation using Einstein's summation convention. No distinction is made between greek and latin letters for indices. They are always understood as 4-dimensional.

To reduce the number of letters used for indices, the same letter may be used more than once when unambiguous, like in

$$g^{gg}\Gamma_{ggc} = \tfrac{1}{2}\, g^{gg} g_{gg,c}\,,$$
$$g^{gg}\Gamma_{agg} = g^{gg}\left(g_{ag,g} - \tfrac{1}{2}\, g_{gg,a}\right)\,.$$

2 Cartesian Conformal Map

Metric Tensor

From a metric tensor, describing a conformal metric and constructed as

$$g_{ab} \;:=\; e^{2\alpha}\,\eta_{ab} \;=\; \begin{bmatrix} \pm e^{2\alpha} & & \\ & \ddots & \\ & & \pm e^{2\alpha} \end{bmatrix} \quad\Leftrightarrow\quad g^{ab} \;:=\; e^{-2\alpha}\,\eta^{ab}\,, \tag{1}$$

we find metric derivatives,

$$\begin{aligned} g_{ab,c} &= \partial_c \, e^{2\alpha} \, \eta_{ab} = 2\alpha_{,c} \, e^{2\alpha} \, \eta_{ab} \\ &= 2 \, E^2 \, \eta_{ab} \alpha_{,c} \, , \end{aligned}$$

and the Christoffel symbol of the first kind,

$$\Gamma_{abc} = \tfrac{1}{2} \left(g_{bc,a} + g_{ac,b} - g_{ab,c} \right) \, .$$

Metric Connection

From that we get the Christoffel symbol of the second kind,

$$\begin{aligned} \Gamma^{a}{}_{bc} &= g^{a\alpha} \Gamma_{\alpha bc} \\ &= \delta^{a}{}_{b} \alpha_{,c} + \delta^{a}{}_{c} \alpha_{,b} - g^{a\alpha} \alpha_{,\alpha} g_{bc} \, , \end{aligned} \tag{2}$$

with the particular contractions

$$\Gamma^{\delta}{}_{\delta c} = n\alpha_{,c} \, , \tag{3}$$

$$\Gamma^{a}{}_{gg} g^{gg} = -(n-2)\alpha_{,g} \, g^{ga} \, . \tag{4}$$

Covariant Derivatives

From (2), the covariant derivative of an arbitrary covector V_a is given by

$$\begin{aligned} \nabla_d V_a &= V_{a;d} = V_{a,d} - V_\gamma \, \Gamma^{\gamma}{}_{ad} \\ &= V_{a,d} - V_a \, \alpha_{,d} - V_d \, \alpha_{,a} + V_g \, \alpha_{,g} \, g^{gg} g_{ad} \, , \end{aligned}$$

so the contracted second covariant derivative, that is, the Laplace operator (or d'Alembert operator, in four-dimensional spacetime) of the logarithmic conformal potential is then

$$\begin{aligned} \nabla_\mu \nabla^\mu \alpha &= \alpha_{;gg} \, g^{gg} \\ &= \left(\alpha_{,gg} + (n-2)\alpha_{,g}\alpha_{,g} \right) \, g^{gg} \, . \end{aligned} \tag{5}$$

Connection Derivatives

From (3) and (4) we get the partial derivatives of the contracted connection,

$$\begin{aligned} \Gamma^{\lambda}{}_{\lambda c,d} &= n \, \alpha_{,cd} \, , \\ \Gamma^{a}{}_{gg,d} \, g^{gg} &= -(n-2) \, g^{a\mu} \, \alpha_{,\mu d} \, . \end{aligned}$$

and their second contractions,

$$\begin{aligned} \Gamma^{\lambda}{}_{\lambda g,g}\, g^{gg} &= n\, \alpha_{,gg}\, g^{gg}\,, && (6)\\ \Gamma^{\lambda}{}_{gg,\lambda}\, g^{gg} &= -(n-2)\, \alpha_{,gg}\, g^{gg}\,. && (7) \end{aligned}$$

So from (6) and (7), the fully contracted connection derivative difference is

$$\left(\Gamma^{\lambda}{}_{\lambda g,g} - \Gamma^{\lambda}{}_{gg,\lambda}\right) g^{gg} = 2\,(n-1)\, \alpha_{,gg}\, g^{gg}\,. \tag{8}$$

Connection Products

From (3) and (4) we get the fully contracted 'straight' connection product,

$$\Gamma^{\lambda}{}_{\lambda\gamma}\, \Gamma^{\gamma}{}_{gg}\, g^{gg} = -n\,(n-2)\, \alpha_{,g}\alpha_{,g}\, g^{gg}\,, \tag{9}$$

while the calculation of the 'crossed' connection product needs a more general calculation from (2),

$$\begin{aligned} \Gamma^{a}{}_{b\gamma}\, \Gamma^{\gamma}{}_{cd} &= \left(\delta^{a}{}_{b}\alpha_{,\gamma} + \delta^{a}{}_{\gamma}\alpha_{,b} - g^{a\rho}\alpha_{,\rho}\, g_{b\gamma}\right) \cdot \left(\delta^{\gamma}{}_{c}\alpha_{,d} + \delta^{\gamma}{}_{d}\alpha_{,c} - g^{\gamma\sigma}\alpha_{,\sigma}\, g_{cd}\right)\\ &= \delta^{a}{}_{b}\left(2\,\alpha_{,c}\alpha_{,d} - \alpha_{,g}\alpha_{,g}\, g^{gg} g_{cd}\right) + \left(\delta^{a}{}_{c}\alpha_{,b}\alpha_{,d} + \delta^{a}{}_{d}\alpha_{,b}\alpha_{,c}\right) - g^{a\alpha}\alpha_{,\alpha}\left(\alpha_{,c}\, g_{bd} + \alpha_{,d}\, g_{bc}\right)\,, \end{aligned}$$

contracting once,

$$\Gamma^{\lambda}{}_{b\gamma}\, \Gamma^{\gamma}{}_{c\lambda} = (n+2)\, \alpha_{,b}\alpha_{,c} - 2\, \alpha_{,g}\alpha_{,g}\, g^{gg}\, g_{bc}\,,$$

and fully contracted,

$$\Gamma^{\lambda}{}_{g\gamma}\, \Gamma^{\gamma}{}_{g\lambda}\, g^{gg} = -(n-2)\, \alpha_{,g}\alpha_{,g}\, g^{gg}\,. \tag{10}$$

From (9) and (10) we get the fully contracted connection product difference,

$$\left(\Gamma^{\lambda}{}_{\lambda\gamma}\, \Gamma^{\gamma}{}_{gg} - \Gamma^{\lambda}{}_{g\gamma}\, \Gamma^{\gamma}{}_{g\lambda}\right) g^{gg} = -(n-1)\,(n-2)\, \alpha_{,g}\alpha_{,g}\, g^{gg}\,. \tag{11}$$

Scalar Curvature

The fully contracted Riemann tensor is now obtained from (11) and (8),

$$\begin{aligned} R &= \left(\left(\Gamma^{\lambda}{}_{\lambda\gamma}\, \Gamma^{\gamma}{}_{gg} - \Gamma^{\lambda}{}_{g\gamma}\, \Gamma^{\gamma}{}_{g\lambda}\right) - \left(\Gamma^{\lambda}{}_{\lambda g,g} - \Gamma^{\lambda}{}_{gg,\lambda}\right)\right) g^{gg}\\ &= -\,(n-1)\left(2\, \alpha_{,gg} + (n-2)\alpha_{,g}\alpha_{,g}\right) g^{gg}\,, && (12) \end{aligned}$$

According to (5), the second partial derivative can be substituted by the covariant derivative,

$$\alpha_{,gg}\, g^{gg} \;=\; \big(\alpha_{;gg} - (n-2)\,\alpha_{,g}\alpha_{,g}\big)\, g^{gg}\,,$$

so (12) can be expressed with covariant derivatives,

$$R \;=\; -\,(n-1)\,\big(\,2\,\alpha_{;gg} - (n-2)\alpha_{;g}\alpha_{;g}\,\big)\, g^{gg}\,. \tag{13}$$

As a corollary follows, that in $(n = 2)$ dimensions curvature of the conformal space is proportional to the Laplacian,

$$R_{(2\mathrm{D})} \;=\; -2\,\alpha_{;gg}\, g^{gg}\,.$$

3 Variations

To do variation of the conformal logarithm function $\alpha(ct, x, y, z)$ over the coordinates ct, x, y, z, the Euler-Lagrange formalism on continuous fields is employed, similar to its use in classical field theory to derive Maxwell's equations from the electromagnetic tensor.

The variational derivative, varying a Lagrangian $\mathcal{L}$ over a function f, is given by

$$\begin{aligned}\delta_f\mathcal{L} \;:=\; \frac{\delta\mathcal{L}}{\delta f} \;&=\; \frac{\partial\mathcal{L}}{\partial f} - \left(\frac{\delta\mathcal{L}}{\delta f_{;a}}\right)_{;a} \qquad \text{(an infinite recursion)}\\ &=\; \frac{\partial\mathcal{L}}{\partial f} - \left(\frac{\partial\mathcal{L}}{\partial f_{;a}}\right)_{;a} + \left(\frac{\partial\mathcal{L}}{\partial f_{;ab}}\right)_{;ab} - \;\cdots\,,\end{aligned}$$

in such a way that the Euler-Lagrange equation reads

$$\delta_f\mathcal{L} \;\overset{!}{=}\; 0\,.$$

Varying the Scalar Curvature

From the Ricci scalar, expressed with covariant derivatives, (13),

$$R \;=\; (n-1)\,\big(-2\alpha_{;gg} + (n-2)\,\alpha_{;g}\alpha_{;g}\big)\, g^{gg}\,,$$

after stripping constant factors, we get an effective Lagrangian density

$$\mathcal{L} \;\overset{\star}{=}\; -2\alpha_{;gg}\, g^{gg} + (n-2)\,\alpha_{;g}\alpha_{;g}\, g^{gg}\,.$$

This Lagrangian does not contain the function α itself, but only the first and second covariant derivatives thereof, $\alpha_{;\mu}$ and $\alpha_{;\mu\nu}$.

So the partial derivative over the function itself vanishes,

$$\frac{\partial \mathcal{L}}{\partial \alpha} \;=\; 0\,,$$

and does not contribute to the results.

Then the partial derivative over the second covariant derivative,

$$\frac{\partial \mathcal{L}}{\partial \alpha_{;\mu\nu}} \;=\; -2\,\delta^a_g\,\delta^b_g\,g^{gg}\,,$$

is constant with respect to the next two covariant derivations,

$$\nabla_a \nabla_b \frac{\partial \mathcal{L}}{\partial \alpha_{;\mu\nu}} \;=\; -2\left(\delta^a_g\,\delta^b_g\,g^{gg}\right)_{;ab} \;=\; 0\,,$$

and so the contribution of the second covariant derivative (the ‘source density’) also vanishes.

Finally of the partial derivative over the first covariant derivative,

$$\frac{\partial \mathcal{L}}{\partial \alpha_{;\mu}} \;=\; 2(n-2)\left(\delta^\mu_g \alpha_{;g}\right) g^{gg}\,,$$

the covariant derivative is non-zero,

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial \alpha_{;\mu}} \;=\; 2(n-2)\alpha_{;gg}\,g^{gg}\,.$$

Field Equation

Again omitting constant factors, the Euler-Lagrange-condition gives

$$\alpha_{;gg}\,g^{gg} \;=\; \Box\,\alpha \;\stackrel{!}{=}\; 0\,, \tag{14}$$

which is a homogeneous Laplace equation, and especially on 4D spacetime is the homogeneous d’Alembert equation, which tells that in $n > 2$ dimensions the ‘source density’ of the conformal logarithm field vanishes identically. In this sense, the field can be said to be ‘source-free’.

Energy Expression

Then putting (14) back into the scalar curvature expression gives a reduced scalar curvature,

$$R \overset{\star}{=} (n-1)(n-2)\,\alpha_{;g}\alpha_{;g}\,g^{gg} \;=\; (n-1)(n-2)\,\nabla\alpha\cdot\nabla\alpha\,, \tag{15}$$

which reads: Mean (scalar) curvature is proportional to the squared magnitude of the gradient of the conformal logarithm function. Thus mean curvature can easily be interpreted as the energy density of the logarithmic conformal field, which is the classical square of the conformal logarithm's gradient vector.

Only in 2D (and trivially in 1D), any conformal metric from a source-free conformal logarithm function gives a flat space; those are the well-known holomorphic and meromorphic functions on the space of complex numbers. The same does not hold true for higher dimensions, $n \geq 3$, and hence on 4D spacetime, where any non-constant conformal metric introduces a curvature.

In dimensions $n > 2$, any conformal function (which is not simply constant) curves space; there is no non-trivial conformal metric which leaves spacetime flat.

Varying the Historical Lagrangian

The historical Hilbert-Einstein action is given as

$$\mathcal{L} \;=\; \sqrt{|g|}R\,,$$

where for the above (1) metric

$$\sqrt{|g|} \;=\; e^{4\alpha}\,.$$

Now investigate the even more generalized ansatz

$$\mathcal{L} \;=\; e^{2k\alpha}\,R\,,$$

so again from the Ricci scalar, expressed with covariant derivatives, (13), and omitting the constant factor $-(n-1)$, an effective Lagrange density is

$$\mathcal{L} \overset{\star}{=} R\,e^{2k\alpha} \;=\; \big(2\alpha_{;gg} - (n-2)\,\alpha_{;g}\alpha_{;g}\big)\,g^{gg}\,e^{2k\alpha}\,.$$

Now the Lagrangian contains the function α inside the exponential $e^{2k\alpha}$, as well as the first and second covariant derivatives, $\alpha_{;\mu}$ and $\alpha_{;\mu\nu}$, so the partial derivative over the function α does not vanish,

$$\frac{\partial\mathcal{L}}{\partial\alpha} \;=\; 2k\,R\,e^{2k\alpha}\,.$$

Then the partial derivative over the second covariant derivative,

$$\frac{\partial\mathcal{L}}{\partial\alpha_{;ab}} \;=\; 2\,\delta^a_g\,\delta^b_g\,g^{gg}\,e^{2k\alpha}\,,$$

and the twice covariant derivation,

$$\nabla_a\nabla_b\frac{\partial\mathcal{L}}{\partial\alpha_{;ab}} \;=\; 2\big(2k\,\alpha_{;gg} + 4k^2\,\alpha_{;g}\alpha_{;g}\big)\,g^{gg}\,e^{2k\alpha}\,,$$

which does not vanish.

Finally of the partial derivative over the first covariant derivative,

$$\frac{\partial\mathcal{L}}{\partial\alpha_{;\mu}} \;=\; (n-2)\,\delta^\mu_g\alpha_{;g}\,g^{gg}\,e^{2k\alpha}\,,$$

the covariant derivative is

$$\nabla_\mu\frac{\partial\mathcal{L}}{\partial\alpha_{;\mu}} \;=\; (n-2)\,\big(\alpha_{;gg} + 2k\,\alpha_{;g}\alpha_{;g}\big)\,g^{gg}\,e^{2k\alpha}\,.$$

Additionally omitting the constant factors $e^{2k\alpha}$, the Euler-Lagrange-condition gives

$$\big((n-2+8k)\,\alpha_{;gg} + 2\,(4k^2+k)\,\alpha_{;g}\alpha_{;g}\big)\,g^{gg} \;\overset{!}{=}\; 0\,.$$

For $k=0$ this gives the same solution as before, (14). For any other choice of $k\neq 0$, the field equation is not homogeneous and thus the field is not source-free. For Hilbert's Lagrangian with $k=2$ in $n=4$ dimensions, the resulting field equation would be

$$\alpha_{;gg}\,g^{gg} \;=\; -2\,\alpha_{;g}\alpha_{;g}\,g^{gg}\,. \tag{16}$$

4 Discussion

In the context of General Relativity, Hilbert chose for the Lagrange density in the 4D case

$$\mathcal{L} \;=\; \sqrt{g}R \;=\; E^4 R\,,$$

to account for the volume element in some way, but this would not give a reasonable wave equation here. Instead, the conformal wave equation arises only when no power of E is multiplied in, which means, the Lagrangian is the scalar curvature (mean curvature, Ricci scalar) itself.

Hence the Langrange density which is proposed as correct instead of the Hilbert-Einstein action is in any number of dimensions the plain Ricci scalar itself without any multiplier,

$$\mathcal{L} \;=\; R\,.$$

Having exercised a purely conformal ansatz here does not mean in general to be restricted to conformal fields. More general problems might involve additional 'tidal' fields and still be solved through the presented Lagrangian.