On Newtonian knot in empty (2+1)-dimensional space-time

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We propose the existence of a topological object, a Newtonian knot, in the framework of an Abelian Chern-Simons gravity with a small positive cosmological constant in empty (2+1)-dimensional space-time. This proposal is based on the idea that the Ricci curvature tensor could consist of a set of curvature components satisfying the non-trivial Hopf maps, leading to topological structures. Working within the Abelian Chern-Simons (first-order) framework, where the dreibein and spin connection are treated as independent fields, we derive the corresponding field equations and present ansatz solutions for both. Our results suggest that the Newtonian knot may serve as a novel topological feature in low-dimensional gravity theories.

Keywords: *(2+1) gravity theory, empty space-time, cosmological constant, Newtonian limit, Abelian Chern-Simons action, knot.*

I. INTRODUCTION

It has been widely believed that topological objects can not exist in linear theories. Topological theories are inherently non-linear[1]. How, then, could a topological object, like a Newtonian knot, exist in the linear theory, such as an Abelian Chern-Simons theory?

It is well known that the general theory of gravitation is identical to a gauge theory[2–5,26]. Cartan gravity makes general relativity similar to a gauge theory.

The formulation of a gravitational knot for a non-Abelian Chern-Simons action in empty (2+1)-dimensional space-time has been proposed[4,5,7–9]. Also, a gravitational knot of the weak-field limit in empty (2+1)-dimensional space-time with a small positive cosmological constant has been constructed[10].

In this article, we propose the existence of a Newtonian knot in the Newtonian limit in empty (2+1)-dimensional space-time formulated as an Abelian Chern-Simons action with a small positive cosmological constant written using the Clebsch variables. The Newtonian limit is related to the weak gravitational field and the objects (e.g. the orbits of planets around the Sun) move very slowly compared to the velocity of light. To the best of our knowledge[1,4,5,7–9,11–17], the formulation of such knot has not been done yet.

We assume that, in analogy to the linearized Ricci curvature tensor in (3+1)-dimensional space-time, the linearized Ricci curvature tensor (with a small positive cosmological constant) is valid in (2+1)-dimensional space-time. The existence of a topological structure in empty three-dimensional space-time gravity is similar to that in Maxwell's theory of a vacuum[1]. What we mean by an empty space-time is a space-time where there is no matter source present and there exist no physical fields except the gravitational field[27]. This gravitational field does not disturb the emptyness, but other fields do[27]. A vacuum is defined as a space without charge and current[28].

Analogous to Maxwell's theory of a vacuum where the field strength tensor could consist of a set of subset fields[1,18], complex scalar fields, we propose that the Ricci curvature tensor (the set of the solutions of Einstein field equations) could consist of a set of curvature components, complex scalar potentials. So, scalar fields in Maxwell's theory are analogous to scalar potential in gravity theory. This set of curvature components, such as a set of subset fields, satisfies the non-trivial Hopf maps. It means that the non-trivial Hopf maps could describe the properties of a set of curvature components.

A set of curvature components is locally equal to the linearized Ricci curvature tensor, i.e. the linearized Ricci curvature tensor can be obtained by patching together a set of curvature components (except in a zero-measure set) but globally different. The difference is global, instead of local, since a set of curvature components obeys the topological quantum condition, but the linearized Ricci curvature tensor does not. The linearized Ricci curvature tensor satisfies the linear Ricci theory, but a set of curvature components satisfies the non-linear Ricci theory. Both, the linearized Ricci curvature tensor and a set of curvature components, satisfy the linear Ricci theory in the case of a weak-field limit where the non-linear Ricci theory reduces to the linear Ricci theory.

II. METRIC PERTURBATIONS AS POTENTIALS

In gravity theory, the linearized metric perturbations take a role as "potentials"[19]. We consider these linearized metric perturbations analogous to a set of curvature components, scalar potentials. Because scalar potentials or scalar fields could be complex, we consider that the linearized metric perturbations could also be complex. In the language of a wave, the linearized metric perturbations could be written as[20]

$$h = \rho(\vec{r},t)\ e^{iq(\vec{r},t)}, \quad h^{*} = \rho(\vec{r},t)\ e^{-iq(\vec{r},t)} \tag{1}$$

where $\rho(\vec{r},t)$ is the amplitude, $q(\vec{r},t)$ is the phase, h^{*} is the complex conjugate of h, i is an imaginary number, $\vec{r}$ is a position vector, t is time. From eq.(1), we take the physical perturbation as its real part[29].

The related (real) vector potential could be written as

$$h_{\mu} = f\ \partial_{\mu} q \tag{2}$$

where the Greek index, μ, denotes space-time coordinates. In the (2+1)-dimensional space-time $\mu = 0, 1, 2$. The amplitude function, f, can be written as below

$$f = -1/\left\{2\pi(1+\rho^{2})\right\} \tag{3}$$

Here f and q are the Clebsch variables[21] or the Gaussian potentials[7,22]. Both, f and q, are scalars.

We will see that this vector potential (2) could be related to, in terms of the Cartan gravity, the gauge fields (the dreibein, the spin connection). Following, we will show that a set of curvature components satisfies the non-trivial Hopf maps.

III. HOPF MAPS

The properties of a set of curvature components or the complex scalar potentials could be described by the non-trivial Hopf maps written below

$$h(\vec{r}),\ h^{*}(\vec{r}) : S^{3} \rightarrow S^{2} \tag{4}$$

where S^{3} and S^{2} denote the three and two-dimensional spheres (space), respectively.

The Hopf maps (4) can be classified in homotopy classes, labeled by the value of the corresponding Hopf indexes, integer numbers, and the topological invariants[1,18]. The other names of the topological invariants are the topological charge, the winding number (the degree of a continuous mapping). The topological charge is metric tensor-independent. It could be interpreted as energy[30].

There exists (one) dimensional space reduction in the Hopf maps (4). We interpret this dimensional reduction as a consequence of the isotropic (well-defined) property of the scalar potential for an infinite r. A set of curvature components consisting of the complex scalar potentials has properties that, by definition, its value for a finite distance, r, depends on the magnitude and the direction of the position vector, $\vec{r}$. Still, for an infinite r, it is well-defined[18] (it depends on the magnitude only). In other words, for an infinite r, the scalar potential is isotropic.

We see that these complex scalar potentials which satisfy the non-trivial Hopf maps (4) are time-independent. Analogous to the time-independent complex scalar fields, this problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[21].

IV. GAUGE POTENTIAL AND GAUGE FIELDS

We interpret the vector potential (2) as the gauge potential which could be decomposed into the gauge fields written below[4,5,7]

$$A_{\mu} = e^{a}_{\mu}\ P_{a} + \omega^{a}_{\mu}\ J_{a} \tag{5}$$

where e^{a}_{μ} and ω^{a}_{μ} are components of the dreibein and the spin connection, respectively. P_{a}, J_{a}, are the generators of translation and Lorentz rotation of ISO(2,1) Poincare group, respectively. The Latin index, a, denotes the local Lorentz index. Eq.(5) shows that the dreibein and the spin connection are treated as independent gauge fields.

The linearized Ricci curvature tensor in the case of the weak-field limit and a small positive cosmological constant, $\Lambda > 0$, $|\Lambda| << 1$ (de Sitter space) can be written as[10]

$$R^{a}_{\nu\rho} = \partial_{\nu}\omega^{a}_{\rho} - \partial_{\rho}\omega^{a}_{\nu} + \frac{\Lambda}{3}\ \varepsilon^{a}{}_{bc}\ e^{b}_{\nu}\ e^{c}_{\rho} \tag{6}$$

where $\varepsilon^{a}{}_{bc}$ is the Levi-Civita symbol which has a role as the structure constants (the structure coefficients)[23]. Eq.(6) is a linear equation in terms of the spin connection.

The extension to the Newtonian limit could be worked by neglecting time derivatives in the linearized Ricci curvature tensor (6), we obtain

$$R^{a}{}_{tj} = -\partial_{j}\omega^{a}_{t} + \frac{\Lambda}{3}\ \varepsilon^{a}{}_{bc}\ e^{b}_{t}\ e^{c}_{j} \tag{7}$$

and

$$R^{a}{}_{jk} = \partial_{j}\omega^{a}_{k} - \partial_{k}\omega^{a}_{j} + \frac{\Lambda}{3}\ \varepsilon^{a}{}_{bc}\ e^{b}_{j}\ e^{c}_{k} \tag{8}$$

where t denotes the time index, $j, k = 1, 2$, denote spatial indices, $\varepsilon^{a}{}_{bc}$ is the Levi-Civita symbol which has a role as the structure constants (the structure coefficients)[23] showing explicitly there exists an interaction between the dreibein.

V. ABELIAN CHERN-SIMONS ACTION

Using the linearized Ricci curvature tensor written in eqs.(7), (8), an Abelian Chern-Simons action could be written as below

$$I_{\rm CS} = \int_{M} \left\{ \varepsilon^{itj}\ e_{ia} \left(-\partial_{j}\omega^{a}_{t} + \frac{\Lambda}{3}\ \varepsilon^{a}{}_{bc}\ e^{b}_{t}\ e^{c}_{j} \right) + \varepsilon^{ijk}\ e_{ia} \left(\partial_{j}\omega^{a}_{k} - \partial_{k}\omega^{a}_{j} + \frac{\Lambda}{3}\ \varepsilon^{a}{}_{bc}\ e^{b}_{j}\ e^{c}_{k} \right) \right\} d^{2+1}x \tag{9}$$

This Abelian Chern-Simons action, (9), is the topological invariant.

VI. FIELD EQUATIONS

The field equations can be derived by applying the variational principle to an Abelian Chern-Simons action, (9), with respect to the dreibein and the spin connection. We obtain the field equations as written below

$$\varepsilon^{ijk}\ \partial_j e_{ia} = 0 \tag{10}$$

$$\varepsilon^{itj}\ (-\partial_j \omega_t^a + \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e_t^b\ e_j^c) + \varepsilon^{ijk}\ (\partial_j \omega_k^a - \partial_k \omega_j^a + \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e_j^b\ e_k^c) = 0 \tag{11}$$

Eqs.(10), (11), are the field equations obtained by varying the spin connection and the dreibein, respectively. Eq.(10) shows that the dreibein satisfies the torsion-free condition. Eq.(11) is analogous to Einstein field equations in an empty (3+1)-dimensional space-time.

We would like to point out that invertibility of the dreibein is required. It means that there exists a well-defined inverse of the dreibein at every point (globally) in the space-time.

VII. NEWTONIAN KNOT AS SOLUTIONS

The Newtonian knot in this context is the special solutions of eqs.(10), (11). It has a consequence that the linearized Ricci curvature tensor, especially the dreibein and the spin connection, as we will see, have non-trivial topology. In other words, we could say that the Newtonian knot has a non-trivial holonomy, anholonomy. What we mean with anholonomy here is after parallel transporting the phase gradient along a closed curve (loop) in a given space-time and then bringing it back to its starting point, the phase gradient has rotated or changed due to the curvature of the space-time.

Analogous to hydrodynamics, eq.(10) is identical to the curl-free velocity, $\Omega = 0$, where the vorticity $\Omega = \varepsilon^{ij}\ \partial_i v_j$[22], v_j is the velocity. We see from (10) that the dreibein is identical to the velocity, and the vorticity is identical to the curvature. Eq.(10) imposes a constraint on the components of the dreibein. The consequence of the curl-free vector field, such as the velocity or the dreibein, is the vector field could be written as the gradient of a scalar function[24]. It implies that the dreibein could be written as

$$e_{ia} = f_e\ \partial_i q_{ea} \tag{12}$$

where f_e is an amplitude function of the dreibein and q_{ea} is the dreibein phase. This formulation of the dreibein (12) is analogous to the vector potential (2), where f_e is analogous to eq.(3).

Mathematically, to ensure that the curl-free vector field can be replaced by the gradient of a scalar function, we should take the scalar function f_e (12) as a constant, so that its derivative vanishes. Analogous to electromagnetism the amplitude is constant in a vacuum, the amplitude is constant in an empty space-time (we assume that vacuum is analogous to an empty space-time), and we could take it has a very small value but non-zero, so that ρ^2 in eq.(3) could be ignored. Then, eq.(3) becomes

$$f_e = -1/2\pi \tag{13}$$

This value of f_e satisfies the mathematical requirement that the scalar function f_e is a constant. A very small but non-zero amplitude combined with the neglect of time derivatives of the spatial components of the spin connection is consistent with the Newtonian limit.

The phase q is mathematically well-defined as long as the amplitude is not equal to zero. Because, if the amplitude is zero then the linearized metric perturbation (1) is also zero. It implies that the phase becomes physically meaningless or undetermined since any phase choice gives the same trivial result.

In an empty space-time where the amplitude is constant and has a very small value but not zero, the phase could be a multivalued function. This multi-valued function of the phase is the consequence of the non-trivial topology of the linearized Ricci structure as we mentioned previously. The multi-valued phase could be written as

$$q_{ea} = m\theta_{ea} + c r_{ea} \tag{14}$$

where m is an integer number, c is a constant. Eq.(14) is the dreibein phase ansatz.

By substituting eqs.(13), (14), into (12), we obtain

$$e_{ia} = -\frac{1}{2\pi}\ \partial_i (m\theta_{ea} + c r_{ea}) \tag{15}$$

Eq.(15) is the dreibein ansatz.

Let us see whether the dreibein ansatz (15) satisfies the solutions of the field equations by substituting eq.(15) into (10), we obtain

$$-\frac{1}{2\pi}\varepsilon^{ijk}\ \partial_j \partial_i (m\theta_{ea} + c r_{ea}) = 0 \tag{16}$$

Eq.(16) shows that the dreibein ansatz satisfies the solutions of the field equations for the dreibein. The zero result is due to the second derivative of an integer and a constant.

Let us analyze the field equations of the spin connection (11). Since the spin connection is also a fundamental field derived from the Abelian Chern-Simons action such as the dreibein, its structure is expected to exhibit similar topological features, making it reasonable to assume a similar gradient-based form such as the dreibein. So, we assume that the solution of eq.(11) is analogous to eq.(12), written below

$$\omega_t^a = f_\omega\ \partial_t q_\omega^a \tag{17}$$

$$\omega_j^a = f_\omega\ \partial_j q_\omega^a \tag{18}$$

$$\omega_k^a = f_\omega \; \partial_k q_\omega^a \tag{19}$$

where f_ω, q_ω^a, are the amplitude function and the phase of the spin connection, respectively. Eq.(17) suggests that even in a Newtonian limit, the system may exhibit slowly evolving topological or geometric phases.

In an empty space-time, such as the amplitude function and phase of the dreibein, we take the value of f_ω as follows

$$f_\omega = -1/2\pi \tag{20}$$

and the multi-valued spin connection phase as

$$q_\omega^a = m\theta_\omega^a + cr_\omega^a \tag{21}$$

By substituting eqs.(20), (21), into (17), (18), (19), we obtain

$$\omega_t^a = -\frac{1}{2\pi}\; \partial_t(m\theta_\omega^a + cr_\omega^a) \tag{22}$$

$$\omega_j^a = -\frac{1}{2\pi}\; \partial_j(m\theta_\omega^a + cr_\omega^a) \tag{23}$$

$$\omega_k^a = -\frac{1}{2\pi}\; \partial_k(m\theta_\omega^a + cr_\omega^a) \tag{24}$$

By substituting eqs.(22), (23), (24), into eq.(11), we obtain

$$\begin{aligned}
&\varepsilon^{itj}\Bigg(\partial_j\partial_t(m\theta_\omega^a + cr_\omega^a) \\
&+ \frac{1}{2\pi}\,\frac{\Lambda}{3}\,\varepsilon^a{}_{bc}\;\partial_t(m\theta_e^b + cr_e^b)\;\partial_j(m\theta_e^c + cr_e^c)\Bigg) \\
&+ \frac{1}{2\pi}\,\varepsilon^{ijk}\Bigg([\partial_k,\partial_j](m\theta_\omega^a + cr_\omega^a) \\
&+ \frac{1}{4\pi^2}\,\frac{\Lambda}{3}\,\varepsilon^a{}_{bc}\;\partial_j(m\theta_e^b + cr_e^b)\;\partial_k(m\theta_e^c + cr_e^c)\Bigg) = 0
\end{aligned} \tag{25}$$

The non-commutativity in eq.(25) could be written as

$$[\partial_k, \partial_j] = f_{kjl}\;\partial_l \tag{26}$$

where f_{kjl} are the non-zero antisymmetric structure coefficients (the structure constants), the non-zero constants. This non-zero commutation relation in eq.(26) arises because space has a non-commutative structure (an intrinsic curvature).

By substituting eq.(26) into (25), we obtain

$$\begin{aligned}
&\varepsilon^{itj}\Bigg(\partial_j\partial_t(m\theta_\omega^a + cr_\omega^a) \\
&+ \frac{1}{2\pi}\,\frac{\Lambda}{3}\,\varepsilon^a{}_{bc}\;\partial_t(m\theta_e^b + cr_e^b)\;\partial_j(m\theta_e^c + cr_e^c)\Bigg) \\
&+ \frac{1}{2\pi}\,\varepsilon^{ijk}\Bigg(f_{kjl}\;\partial_l(m\theta_\omega^a + cr_\omega^a) \\
&+ \frac{1}{4\pi^2}\,\frac{\Lambda}{3}\,\varepsilon^a{}_{bc}\;\partial_j(m\theta_e^b + cr_e^b)\;\partial_k(m\theta_e^c + cr_e^c)\Bigg) = 0
\end{aligned} \tag{27}$$

Eq.(27) suggests that the non-commutative nature of space is induced by the presence of a cosmological constant. In other words, the structure constants, f_{kjl}, which determine the non-commutativity of spatial derivatives, could be a consequence of a non-zero cosmological constant. This aligns with the idea that a curved space (e.g., with constant curvature due to the cosmological constant) naturally leads to a modification of coordinate algebra, making the space non-commutative.

VIII. DISCUSSION AND CONCLUSION

Eq.(10) says that the dreibein components are constant, ensuring that the space-time remains smooth and torsion-free in local space-time, meaning it does not exhibit rotational "twists" in local space-time. This implies that the space is locally flat, at least in terms of the dreibein components themselves. It does not necessarily imply that there is no curvature in the global space-time. The curvature can still exist if the spin connection and cosmological constant are non-zero. The spin connection encodes how the local frames (associated with the dreibein) are "twisting" or "rotating" with respect to each other as we move through space-time.

Eq.(11) shows that the space variation of the time component of the spin connection is proportional to the curvature induced by the cosmological constant and the dreibein configuration. The non-zero spin connection and the cosmological constant together contribute to the curvature of space-time. Eq.(11) can be interpreted as describing how the curvature of the spin connection is influenced by the cosmological constant and the dreibein configuration.

The existence of a multi-valued phase is crucial in defining the topological properties of the system, particularly those associated with anholonomy. If the phase is single-valued then the space-time does not support the existence of the topological object which in our case is the Newtonian knot.

Empirical or observational evidence supporting the existence of the Newtonian knot in (2+1)-dimensional empty space-time is ensured by the formal equivalence between the Newtonian knot and the electromagnetic knot in vacuum Maxwell theory, where knot solutions are known to exist.

The existence of the Newtonian knot in (2+1)-dimensional empty space-time with the small positive cosmological constant (de Sitter space-time) does not support the wide belief[25,26] that there exists no Newtonian limit in (2+1)-dimensional space-time. This work could be extended to the Newtonian limit in (2+1)-dimensional empty space-time with the small negative cosmological constant (anti-de Sitter space-time).

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[1] Antonio F. Ranada, *Topological electromagnetism*, J. Phys. A: Math. Gen. **25** (1992) 1621-1641.
[2] Chen Ning Yang, *Topology and Gauge Theory in Physics*, International Journal of Modern Physics A, Vol. 27, No. 30 (2012) 1230035.
[3] Y. M. Cho, *Gauge theory, gravitation, and symmetry*, Physical Review D, Volume 14, Number 12, 15 December 1976. Y. M. Cho, *Einstein Lagrangian as the translational Yang-Mills Lagrangian*, Physical Review D, Volume 14, Number 10, 15 November 1976.
[4] Edward Witten, *2+1 Dimensional Gravity as an Exactly Soluble System*, Nuclear Physics **B** 311 (1988) 46-78.
[5] Bastian Wemmenhove, *Quantisation of 2+1 dimensional Gravity as a Chern-Simons theory*, Thesis, Instituut voor Theoretische Fysica Amsterdam, 2002.
[6] Steven Carlip, *Quantum Gravity in 2+1 Dimensions*, Cambridge University Press, 2003.
[7] Roman Jackiw, *Diverse Topics in Theoretical and Mathematical Physics*, World Scientific, 1995.
[8] Roman Jackiw, *Lower Dimensional Gravity*, Nuclear Physics **B** 252 (1985) 343-356.
[9] Wouter Merbis, *Chern-Simons–like Theories of Gravity*, Ph.D. Thesis, University of Groningen, 2014.
[10] Miftachul Hadi, Suhadi Muliyono, *A weak gravitational knot in (2+1)-dimensional empty space-time*, OSF, 2024, and all references therein, https://osf.io/preprints/osf/vj8pc.
[11] A. V. Crisan, C. R. L. Godinho, I. V. Vancea, *Gravitoelectromagnetic knot fields*, https://arxiv.org/pdf/2103.00217, 2021.
[12] Y. M. Cho, Seung Hun Oh, Pengming Zhang, *Knots in Physics*, International Journal of Modern Physics A, Vol. 33, No. 07, 1830006 (2018).
[13] Y. M. Cho, Franklin H. Cho and J.H. Yoon, *Vacuum decomposition of Einstein's theory and knot topology of vacuum space-time*, Class. Quantum Grav. **30** (2013) 055003 (17pp).
[14] Michael Atiyah, *The Geometry and Physics of Knots*, Cambridge University Press, 1990.
[15] Ji-rong Ren, Ran Li, Yi-shi Duan, *Inner topological structure of Hopf invariant*, https://arxiv.org/abs/0705.4337v1, 2007.
[16] Yi-shi Duan, Xin Liu, Li-bin Fu, *Many knots in Chern-Simons field theory*, Physical Review D **67**, 085022 (2003).
[17] Jorge Zanelli, *Chern-Simons Gravity: From 2+1 to 2n+1 Dimensions*, Brazilian Journal of Physics, Vol. 30, No.2, June 2000.
[18] Antonio F. Ranada, *A Topological Theory of the Electromagnetic Field*, Letters in Mathematical Physics **18**: 97-106, 1989.
[19] James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Pearson, 2014.
[20] Miftachul Hadi, *Knot in weak-field geometrical optics*, OSF, https://osf.io/e7afj/, 2023, and all references therein.
[21] A. F. Ranada, A. Tiemblo, *A Topological Structure in the Set of Classical Free Radiation Electromagnetic Fields*, https://arxiv.org/abs/1407.8145, 29 Jul 2014.
[22] Roman Jackiw, *Lectures on Fluid Dynamics*, Springer, 2002.
[23] Mark Srednicki, *Quantum Field Theory*, Cambridge University Press, 2007.
[24] B. Hague, *An Introduction to Vector Analysis for Physicists and Engineers*, Springer, 1970. Wolfram MathWorld, *Curl*, https://mathworld.wolfram.com/Curl.html.
[25] John D. Barrow, A. B. Burd, David Lancaster, *Three-dimensional classical spacetimes*, Class. Quantum Grav. **3** (1986) 551-567.
[26] S. Carlip, *The (2+1)-Dimensional Black Hole*, arXiv:gr-qc/9506079v1 29 Jun 1995.
[27] P. A. M. Dirac, *General Theory of Relativity*, John Wiley & Sons, 1975.
[28] Richard P. Feynman, Robert B. Leighton, Matthew Sands, *The Feynman Lectures on Physics: The Electromagnetic Field*, Addison-Wesley, 1965.
[29] J. F. Nye, *Natural Focusing and Fine Structure of Light*, IOP Publishing Ltd, 1999.
[30] Miftachul Hadi, Hans Jacobus Wospakrik, *SU(2) Skyrme Model for Hadron*, https://arxiv.org/abs/1007.0888, 2010. Miftachul Hadi, Irwandi Nurdin, Denny Hermawanto, *Analytical Analysis and Numerical Solution of Two Flavours Skyrmion*, https://arxiv.org/abs/1006.5601, 2010.
[31] Li-Bin Fu, Yi-Shi Duan, Hong Zhang, *Evolution of the Chern-Simons vortices*, Physical Review D, **61**, 045004.
[32] Yi-Shi Duan, Xin-Hui Zhang, Li Zhao, *Novel Topological Invariant in the* $U(1)$ *Gauge Field Theory*, arXiv: hep-th/0703165, 19 Mar 2007.