MULTIVARIATE CIRCLE OF PARTITIONS AND THE SQUEEZE PRINCIPLE

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Abstract. The goal of this paper is to extend the squeeze principle to circle of partitions with at least two resident points on their axes.

1. **Introduction**

Let $\mathbb{A} \subset \mathbb{N}$ then we call the set

$$\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}_i) := \left\{ [x_1], [x_2], \ldots, [x_h] \;\middle|\; x_i \in \mathbb{A}_i,\ n = \sum_{i=1}^{h} x_i \right\}$$

a **multivariate** circle of partition generated by $n \in \mathbb{N}$ with base **regulators** $\bigotimes_{i=1}^{h} \mathbb{A}_i$ the h-fold direct product of the sets $\mathbb{A}_i$. We call members of the multivariate circle of partitions multivariate points. We denote the **weight** of each points as $||[x_i]|| := x_i \in \mathbb{A}_i$ and for the corresponding weight set of the multivariate circle of partitions

$$||\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}_i)|| := \{(x_1, x_2, \ldots x_h) \in \bigotimes_{i=1}^{h} \mathbb{A}_i \mid \sum_{i=1}^{h} x_i = n\}.$$

We denote $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$ as an **axis** of the multivariate circle of partitions $\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}_i)$ if and only if $x_i \in \mathbb{A}_i$ for each $1 \leq i \leq h$ and

$$n = \sum_{i=1}^{h} x_i.$$

We say the axis points $[x_i]$ for each $1 \leq i \leq h$ are axis **residents**. We do not view the axis as any different among other axis of the form $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$ up to the rearrangements of its residents points. In special cases where the points

$$[x_k] \in \mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}_i)$$

is such that $hx_k = n$, then we call $[x_i]$ the **center** of the multivariate circle of partitions. If it exists, then we call it as a **degenerated axis** $\mathbb{L}_{[x_k]}$ in comparison to the

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real axes $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$, where not all of the weights x_i can be equal. We denote the assignment of an axis $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$ to the multivariate CoP $\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ as

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}\ \hat{\in}\ \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i) \text{ which means } [x_1],[x_2],\ldots,[x_h]\in\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$$

with

$$n=\sum_{i=1}^{h}x_i$$

for a fixed $n\in\mathbb{N}$ with $x_i\in\mathbb{A}_i$ for each $1\leq i\leq h$ or vice versa and the number of real axes of the multivariate circle of partitions as

$$\nu(n,\bigotimes_{i=1}^{h}\mathbb{A}_i):=\#\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}\ \hat{\in}\ \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)\mid x_i\neq x_j\}$$

for all $1<i<j\leq h$.

In the special case where we fix $h=2$ and take $\mathbb{A}_i=\mathbb{A}\subset\mathbb{N}$, then we obtain the circle of partitions

$$\mathcal{C}(n,\mathbb{A}):=\{[x]\mid x,n-x\in\mathbb{A}\}$$

and the corresponding counting function for the axes set

$$\nu(n,\mathbb{A}):=\#\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\mid x\neq y\}.$$

This structure was studied extensively in [1] in the case where one allows just two axis points on their axes. The squeeze principle is the statement

Theorem 1.1 (The squeeze principle)**.** *Let $\mathbb{B}\subset\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}(m,\mathbb{B})$ and $\mathcal{C}(m+t,\mathbb{B})\neq\emptyset$ for $t\geq 4$. If there exists $\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(m+t,\mathbb{M})$ with $x\in\mathbb{B}$ and $x<y$ such that*

$$y>w:=\max\{u\in||\mathcal{C}(m,\mathbb{M})||\mid u\in\mathbb{B}\}>m-x, \tag{1.1}$$

then there exists $\mathcal{C}(s,\mathbb{B})\neq\emptyset$ such that $m<s<m+t$.

Indeed it has found some unexpected applications to **Goldbach**-type problems and has been applied to study additive prime number problems requiring certain partitions into certain subsets of the positive integers [2]. The power of this principle allows one to exhaust any interval of the form $[n,n+t]$ for $t\geq 4$ in way to conclude that all even numbers in this interval can be written as the sum of two prime numbers. Much more generally the set may be extended to a general subset of the integers and it mostly suffices to check if the underlying conditions of the principle are all satisfied in order to run this test. The squeeze principle (see [2, 4]) has an alternate version which is also valid when one allows a complex base set. We restate it here without proof as a preliminary

Lemma 1.2 (The squeeze principle)**.** *Let $\mathbb{B}\subset\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{M}})$ with $t\geq 4$ be non–empty cCoPs with integers n,t,s of the same parity. If there exist an axis $\mathbb{L}_{[v_1],[w_1]}\ \hat{\in}\ \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ with $w_1\in\mathbb{C}_{\mathbb{B}}$ and an axis $\mathbb{L}_{[v_2],[w_2]}\ \hat{\in}\ \mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{M}})$ with $v_2\in\mathbb{C}_{\mathbb{B}}$ such that*

$$\Re(v_1)<\Re(v_2) \text{ and } \Re(w_1)<\Re(w_2) \tag{1.2}$$

then there exists an axis $\mathbb{L}_{[v_2],[w_1]}\ \hat{\in}\ \mathcal{C}^o(n+s,\mathbb{C}_{\mathbb{B}})$ with $0<s<t$. Hence $\mathcal{C}^o(n+s,\mathbb{C}_{\mathbb{M}})$ is also not empty.

2. The generalized squeeze principle and applications

We obtain an analogous versions of the **squeeze** principle in the setting of axes of circle of partitions with at least two resident points.

Lemma 2.1 (Generalized squeeze principle)**.** *Let* $\mathbb{A} \subset \mathbb{N}$ *with* $\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$ *for a fixed* $n \in \mathbb{H} \subseteq \mathbb{N}$*. If* $t \in \mathbb{N}$ *is such that* n *and* $n+t$ *are not consecutive integers in* $\mathbb{H}$ *and there exists an axes*

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n+t, \bigotimes_{i=1}^{h} \mathbb{N})$$

with $x_i \in \mathbb{A}$ *for all* $1 \leq i \leq h-1$ *and* $x_i < x_h$ *for all* $1 \leq i \leq h-1$ *such that*

$$x_h > w := \max\{u \in ||\mathcal{C}(n, \mathbb{N})|| \mid u \in \mathbb{A}\} > n - \sum_{i=1}^{h-1} x_i$$

then there exists

$$\mathcal{C}(s, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$$

for $n < s < n+t$ *with* $s \in \mathbb{H}$*.*

Proof. We note that from the hypothesis

$$x_h > w := \max\{u \in ||\mathcal{C}(n, \mathbb{N})|| \mid u \in \mathbb{A}\} > n - \sum_{i=1}^{h-1} x_i$$

we can write

$$n = w + (n-w) < w + \sum_{i=1}^{h-1} x_i < \sum_{i=1}^{h} x_i = n+t$$

for $x_i < x_h$ for all $1 \leq i \leq h-1$. Clearly $w \in \mathbb{A}$ with $x_i \in \mathbb{A}$ for $1 \leq i \leq h-1$ so that there exists an axis

$$\mathbb{L}_{[w],[x_1],\ldots,[x_{h-1}]} \hat{\in} \mathcal{C}(s, \bigotimes_{i=1}^{h} \mathbb{A})$$

with $n < s < n+t$ and $s \in \mathbb{H}$, since $n, n+t$ are not consecutive in $\mathbb{H}$. This means that s can be written as the sum of h (not all possibly distinct) elements of $\mathbb{A} \subset \mathbb{N}$. □

We show how this principle can be applied to solve **Goldbach**-type problems with h summands for $h \geq 2$. This is a generalization of the technique we devised in our previous paper [2].

Theorem 2.2 (The partition law)**.** *Let* $\mathbb{A} \subset \mathbb{N}$ *and suppose* $\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$ *for infinitely many* $n \in \mathbb{H} \subseteq \mathbb{N}$*. If for each* $t \in \mathbb{N}$ *such that* $n, n+t$ *are not consecutive in* $\mathbb{H}$ *there exists at least an axis*

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n+t, \bigotimes_{i=1}^{h} \mathbb{N})$$

with $x_i \in \mathbb{A}$ *for all* $1 \leq i \leq h-1$ *and* $x_i < x_h$ *for all* $1 \leq i \leq h-1$ *such that*

$$x_h > w := \max\{u \in ||\mathcal{C}(n,\mathbb{N})|| \mid u \in \mathbb{A}\} > n - \sum_{i=1}^{h-1} x_i$$

then there are multivariate circles of partitions with the property

$$\mathcal{C}(s, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$$

for all $s \geq k$ *for a fixed* $k \in \mathbb{N}$ *with* $s \in \mathbb{H}$*, which means every number in* $\mathbb{H} \subset \mathbb{N}$ $\geq k$ *can be written as the sum of* h *elements (not all possibly distinct) of* $\mathbb{A}$*.*

Proof. Suppose that $\mathbb{A} \subset \mathbb{N}$ and let k be the smallest number in $\mathbb{H} \subseteq \mathbb{N}$ such that $\mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$, which means k is the smallest number in $\mathbb{H}$ such that it can be written as the sum of h elements in $\mathbb{A} \subset \mathbb{N}$. Let us choose $t_o \in \mathbb{N}$ such that $k, k+t_o$ are not consecutive in $\mathbb{H}$, then by the hypothesis there exists at least an axis

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k+t_o, \bigotimes_{i=1}^{h} \mathbb{N})$$

with $x_i \in \mathbb{A}$ for all $1 \leq i \leq h-1$ and $x_i < x_h$ for all $1 \leq i \leq h-1$ such that

$$x_h > w := \max\{u \in ||\mathcal{C}(k,\mathbb{N})|| \mid u \in \mathbb{A}\} > k - \sum_{i=1}^{h-1} x_i.$$

It follows from Lemma 2.1 there exists an $s \in \mathbb{N}$ with $k < s < k+t_o$ such that

$$\mathcal{C}(s, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$$

which means $s \in \mathbb{H}$ can be written as the sum of h (not all) possibly distinct elements of $\mathbb{A}$. Let us consider the sub-intervals $[k,s]$ and $[s,k+t_o]$. If $k < s := k+t_1$ are not consecutive in $\mathbb{H}$ then there exists at least an axis

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k+t_1, \bigotimes_{i=1}^{h} \mathbb{N})$$

with $x_i \in \mathbb{A}$ for all $1 \leq i \leq h-1$ and $x_i < x_h$ for all $1 \leq i \leq h-1$ such that

$$x_h > w := \max\{u \in ||\mathcal{C}(k,\mathbb{N})|| \mid u \in \mathbb{A}\} > k - \sum_{i=1}^{h-1} x_i.$$

It follows from Lemma 2.1 there exists some $u \in \mathbb{H}$ with $k < u < k+t_1$ such that

$$\mathcal{C}(u, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$$

which means $u \in \mathbb{H}$ can also be written as the sum of h (not all) possibly distinct elements of $\mathbb{A}$. We can similarly iterate this argument on the intervals $[k,u],[u,s],[s,k+t_o]$ so far as there exists an element of $\mathbb{H}$ in any of the intervals. By virtue of the hypothesis the $t \in \mathbb{N}$ can be chosen arbitrarily such that $k, k+t$ are not consecutive in $\mathbb{H}$ and with the existence of at least an axis

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k+t_o, \bigotimes_{i=1}^{h} \mathbb{N})$$

with $x_i \in \mathbb{A}$ for all $1 \leq i \leq h-1$ and $x_i < x_h$ for all $1 \leq i \leq h-1$ such that

$$x_h > w := \max\{u \in ||\mathcal{C}(k,\mathbb{N})|| \mid u \in \mathbb{A}\} > k - \sum_{i=1}^{h-1} x_i.$$

The iterative arguments can be extended to all elements of $\mathbb{H}$ under the assumption that

$$\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset$$

for infinitely many $n \in \mathbb{H} \subseteq \mathbb{N}$. This means that every element $n \in \mathbb{H}$ can be written as the sum of h elements of $\mathbb{A}$, not all distinct. ☐

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