Stringy Motivic Spectra II: Higher Koszul Duality

Ryan J. Buchanan

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Dedicated to Derik Monroe. Thanks for listening to my rambling.

Abstract

This is a rendition of [2]. We study stringy motivic structures. This builds upon work dealing with $\mathbb{F}_p$-modives for a suitable prime p. In our case, we let p be a long exact sequence spanning a path in a pre-geometric space. We superize a nerve from our previous study.

0.0 Prologue

Note: If there any typos in this memo, kindly let me know! *rjbuchanan2000@gmail.com*

Let $S^{\infty}_{/s}$ be an infinite slice tower. Then, in the following burger diagram:

$$\begin{tikzcd}
S^0_{/s} \arrow[r] \arrow[d] \arrow[rrrr, bend left, "\mathcal{N}^1_{kosz}"] & ... \arrow[r] & S^n_{/s} \arrow[r] \arrow[d] & ... \arrow[r] & S^{\aleph_0}_{/s} \arrow[d] \\
\Omega S^0_{/s} \arrow[rrrr, bend right, leftrightarrow, "\mathcal{N}^{-1}_{kosz}"'] & & \Omega S^n_{/s} \arrow[ll, squiggly] & & \Omega S^{\aleph_0}_{/s} \arrow[ll, squiggly]
\end{tikzcd}$$

$\aleph_0$ is the zero object of the diagram $S^{\aleph_0}_{/s} \longrightarrow S^{\infty}_{/s}$. The nerve

$$\mathcal{N}^{\pm 1}_{Kosz}$$

is a regular cardinal in the field $\mathbb{K}$ which is spanned by the collection of transition maps in the below Ω-spectra. Denote by

$$\mathcal{N}^{\pm 1}_{/\tau \leq \infty}$$

the Koszul complex associated with the nerve by the flag

$$Fl_0 \sqcup^{\infty}_{i} \mathcal{N}^{\infty}_{/\tau = i}$$

which is the colimit

$$\underset{ni\longrightarrow\infty}{Fl_0} \quad n \in \mathbb{N}$$

Definition 0.1. *By Ω-spectrum, we mean a collection of deloopings*

$$\Sigma\omega_{i\longrightarrow 1} \;\; i \in [0,1]$$

acting upon a stabilizer n times; n being the degree of the Koszul nerve which strikes both at the source and target. This means that the isotropy group of the point realized by the contraction around the target has an isotropy group of order n as well. Thus, we are given the following sequence:

$$\mathscr{G} \supset ... \longrightarrow n \longrightarrow * \longrightarrow n^+ \longrightarrow ...$$

with $n = sup(\mathbb{K})$ is the maximal idea of the field $\mathbb{K}$, and where n_+ is given by fusion with an additive map.

Recall [1] that the Koszul dual of a space $\mathscr{S}$ is given by

$$H^*(BA) \cong Ext^*_{BA}(k,k)$$

with $(k,k) \in A \times \mathbb{K}^\circ$ and BA the classifying space of some immersed subalgebra A of the motivic spectrum.

Remark 0.1. *The Koszul nerve should be thought both to be a fibrant object, and a motivic one. It is effectively geometric as well, in the sense that*

$$\mathcal{N}^{eff} \simeq \; \mathcal{N} \setminus \mathfrak{shad}$$

Contents

1 Superization of the nerve

Let $A^{\pm}$ be a superalgebra. Let us impose the following condition:

$$\mathfrak{a} \in A^{\pm} = \begin{cases} + & \text{if } \mathfrak{a} \notin \mathfrak{shad} \\ - & \text{if } \mathfrak{a} \in \mathfrak{shad} \end{cases}$$

Therefore,

Proposition 1.1. $\mathscr{N}^{eff} = A^{+}$

1.1 Preliminary remarks

I would like to paint a rather geometric picture here. Imagine a light-cone, $\mathbb{L}^4$, which is stratified into positive and negative (odd or even) parts, such that, for all $\mathfrak{a}$, there is $\mathfrak{a}^{-} \vdash \mathfrak{a}^{+}$. Our nerve here is

$$\mathscr{N} \cong \Delta(\mathbb{L}^4)$$

and is closed, symmetric monoidal.

Let $\mathscr{PT}$ be a potential theory for a holomorphic space. The question of this article is the following: "what is $\mathscr{PT}(\mathscr{N})$, and how does it differ from $\mathscr{PT}(\mathscr{N}^{eff})$?" For the purposes of easing this question, I would like to reduce this question to the familiar special case: "let $\mathscr{N}^{eff}\,\langle W\rangle\, A^{-}$, where W is a bordism. Assume that the bordism is an EPR bridge. What are the transport properties of W?

To understand the physical side of this question, we need to induce some sort of torsion into $\mathscr{PT}$, such that

$$\tau(\mathfrak{tors} \hookrightarrow \mathscr{PT}) = 1$$

is a proper truth value in the meta-theory. We would want some families of maps $\rho_{\mathfrak{a}} : \tau(\mathfrak{a}^{\pm}) \longrightarrow Top$, and examine each possible gradation over τ individually. In order to make this problem tractable, we specify which τ-values are admissible by imposing an ultrafilter on a field. For shadow objects, this might involve a choice of functor sending the object to the desired admissible category. For example, one could select a good motivic representation of $\mathbb{F}_p \mapsto \mathbb{K}_{/r}$ to be the qualifying criterion for an object to be embedded flatly into our topological space.

In general, a Reidemeister move on the boundary of a lightcone induces a torsor in the bi-category $\mathfrak{sch}_2$ of 2-schemes, which consists of all operations $\mathfrak{sch} \times \mathfrak{sch}$, such that $\mathfrak{sch}^2 = \mathfrak{sch}_2$. The functors of this category are pre-geometric "quasi-translation" functors which act microlocally on sheaves via operators which act on residues of the ground field. By ground field, we mean the field whose elements are the chosen basis of pointed spaces generated by blow-ups of varieties on the sheaf.

1.2 Calculus with balls

Let B be an open ball, and $\bar{B}$ a closed ball, denoting by B° the group completion (closure) of B. In our model, the openness of a ball is not determined locally. As an example, the collar of the EPR bridge connecting balls is the closure of some hole or particle, but the ball itself is open in the total space. The non-locality of the ball in this case means a real-world, bonafide physical partical non-locality. This means that each particle is locally closed, but *stably speaking*, this closure is not transitive under a map $B \longrightarrow B \cup B^{\circ}$.

The corresponding notion of a *Reidemeister* move on the boundary of a lightcone here is the corresponding *Reidemeister* move on the boundary of $\bar{B}$, which is locally a 5-brane, $\mathcal{B}^5_{\mathcal{A}_d\mathcal{S}}$. The wall-crossing morphisms [2]

$$\mathfrak{Walls} : \mathfrak{D}^m \longrightarrow int(\mathfrak{D}^m)$$

is used for discretely traversing an element in the isotropy group of a point on the infinitesimally thickened boundary of $B^{\circ} \in \mathbb{D}^m$, where here m can be any chosen sup-pole. The Sati-Schreiber tadpole cancellation [3] occurs at this point, but it is not yet a genuine fixed point without the supplement of at least an orbifold structure. This is why the induction of an equivariant cohomotopy theory becomes important.

Locally, some small structures may appear discrete, but microlocally appear quite smooth. Such is the case as when an object in the one object category of spt_* is blown-up into a lens space. Keep in mind that most of the familiar specialization functors terminate in spt_*, so that we have:

$$spt_* \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p \supset K' \longrightarrow ... \longrightarrow spt_*$$

This makes this a very useful, and quite fundamental, object for studying *GUT baryogenesis*. The unwrapping of a motive over a brane results in the annihilation of $\mathfrak{Walls}$.[1] Specifically, every weighted object in $\mathfrak{Walls}$ is sent to the empty set via the deletion of a single point.

$$spt_* \setminus * \longrightarrow \{\emptyset\}$$

Imagine that this point is the crepant resolution of an orbifold $\mathcal{O}$. This functor kills the homotopy type of some distinguished subset $\mathcal{S}^n \subset \mathcal{O}$ via a Postnikov tower:

$$\mathcal{O}_{\tau \leq n} \cong Spt_{*_0} \longrightarrow ... \longrightarrow Spt_{*_n}$$

where

$$\mathcal{O}_{\tau \leq n} \cong inf(\Delta) \longrightarrow sup(\Delta)$$

Applying the *etale realization* functor to every slice in $\mathscr{O}$ results in the annihilation of a quasi-periodic object of the ω-spectrum $\omega\mathscr{O} \sqsubset \Omega\mathscr{O}$.

[1] I had to restrict myself from entering the pun: "breaking the 4th $\mathfrak{Walls} = \not{f}(\mathfrak{W}_4)$. Terrance Tao says not to make notation too cheeky, however.

$$\begin{tikzcd} \delta_{i\leq k}(\mathscr{O}) \arrow[r] \arrow[r, bend left, dashed, "+"] & \delta_\infty(\mathscr{O}) \arrow[l, bend left, dashed, "-"] \end{tikzcd} \qquad \simeq \qquad spt_* \longrightarrow ... \longrightarrow spt_*$$

Remark 1.1. *The above diagram depics the spin of a grvitino, which is 2/3. This is the ind-object of the loop $q_\ell : spt_* \longrightarrow ... \longrightarrow spt_*$. We can write δ_{in} and δ_{out} for the initial and terminal object in the above sequence. Our survey then reduces to studying the maps $\ell : (\delta_{in} \overset{\Sigma}{\mapsto} \delta_{out}) \in \mathbb{L}^4$. For simplicity here,*

$$\Sigma = \cup_0^n \delta_i(\tilde{x})$$

where $\Sigma \in Man$ and $\delta_i(\tilde{x}) \subset \delta_{i<k}\mathscr{O}$. We will regard each $\delta_{i<k}\mathscr{O}$ as Hermitian.

Definition 1.1. *Let Bun_{C^k} be a C^k-bundle. By this, we mean that connections in $End(Bun_{C^k})$ form k-maps $\delta_i(\tilde{x}) \overset{k}{\longmapsto} \delta_{i+k}(x)$.*[2] *The subscript on $\delta_\bullet$ is essentially the level of the map $i \mapsto i+k$.*

Remark 1.2. *The above definition assumes $End(Bun_{C^k})$ yields only pointed sets. With respect to a derivative over time, a C^k space will always yield*

$$\partial^{k-1} n \sim \varepsilon \sim \pi_0(\mathfrak{x}) \simeq x_0$$

Let each x_0 have a $\mathbb{Z}_2$-grade. Then, one obtains

$$E/\mathfrak{H} = \sum_{i=0}^{1} x_i^{\pm}$$

where $\mathfrak{H}$ denotes the unit quaternion, and E is a presheaf of motives over $\mathbb{K}$.

Let E be the principle energy number of a photon, and $\mathfrak{H}$, again, the unit quaternions. Then, E splits once into each of its principle hypercharge directions. Since we are working with pointed spaces, each hypercharge direction takes a motivic object and transforms it into a point via the map

$$E/\mathfrak{H} \longmapsto *$$

For a hyperkaehler manifold, this can be used as a model of a wormhole between pathwise disjoint, but non-trivially globally simply connected towers of sums. The Ω-spectrum of $\mathfrak{H}$ encodes cohomological data about the moment up of schemes in the sheaf E. Specifically, we obtain information (in Bredon cohomology) about a G-CW complex. If we work with a $Sl_3\mathbb{Z}$-local parameter space, we gain a rational index of some open fiber frame. See [4] for more information. Our classifying space is $B\Gamma^q$, a twist on the classical Haefliger space. Observe that for two neighborhoods

$$(\mathcal{U}(x), \mathcal{U}(y)) \in B\Gamma^q$$

[2] The above definition greatly reduces the complexity of C^k-spaces for the k-theoretically minded.

we have a codimension 2 foliation, which manifests itself covariantly as a bordism. There is a superconformal stringy spectrum

$$\Omega^2 B^p \Gamma^q$$

which we will discuss later. Here,

$$\Omega^2(\bullet) = \Omega^1(\bullet) \cup L\Omega^1(\bullet')$$

where each $\bullet$ is a monopole.

We have the following correlation:

$$(q_\ell^n \propto \Omega^n) = Cor_n(q_\ell, \Omega)$$

with $n \in Sl_3\mathbb{N}$, which is a concrete instantiation of a page of a Segal/Atiya/Hirzebruch spectral sequence. Ω^n can be made by tensoring q_ℓ^n with a trivial $O_{\tilde{X}}$-module, where $\tilde{X}$ is a Koszul complex. Recall that a Koszul complex is a diagram with commutative pushouts

$$\begin{tikzcd}
 & T^1_{/c} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
T^1_{/c} \arrow[dr, leftrightarrow] & T^0_{/c} \arrow[u] \arrow[l] \arrow[r] \arrow[d] & T^1_{/c} \arrow[dl, leftrightarrow] \\
 & T^1_{/c} &
\end{tikzcd}$$

sending each n to n_+, where

$$n_+ = n \vee \amalg_i Sp(n)$$

which smashes a copy of n with an $\mathbb{A}^q$-local structure. Every germ that survives this operation is sent to an associative additive map by $\xmapsto{+\varepsilon}$. In our case, we have normalized ε to 1, inducing a cycle on $End(n \cup_{)0,1(} n_+) \subset \Delta \supset n \times n_+$ This means $n \times n_+$ is nullhomotopic, or in other words the Euclidean realization is nilpotent.

2 Spectral fibers

Our main result is the following:

$$\ell^q \in fib(q, p)$$
$$\simeq \ell^{p,q} - p^{ad}$$
$$\simeq \Omega^q(p)$$

where p is a long exact sequence of motives

$$p : ... \longrightarrow Mot/\mathbb{K} \longrightarrow ...$$

and where $\ell^q = X_{p_q}$ is a fiber spectrum of a period-preserving transition map, and where $p_q = Mot/\mathbb{K}'$ for some $\mathbb{K}$'.

Spectra are not necessarily equipped (by default) with a metric structure, but they may be assigned one by a representative topological invariant of a pointed space $\mathcal{E}_*$. That is to say, for $\mathfrak{o} \in \mathcal{E}_*$, there is a corresponding metric μ, where $\mathfrak{o}$ is the evaluation map:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathfrak{o}=ev(X)"] & X' \arrow[d, "\mathfrak{o}=ev(X')"] \\ x \in X \arrow[r] & x \in X' \end{tikzcd}$$

sending each space to its pointed one-object category along the nerve $\mathcal{N}^{\varepsilon}_{Kosz}$, with ε being the canonical group completion:

$$\mathbb{X} \implies \mathbb{X}^{\circ}$$

where $\mathbb{X}$ is given by the pre-ordered set

$$\mathbb{X} = \begin{cases} Set_{\mathbb{K}^{\circ}} & \mathbb{K}^{\circ} \in q_{\ell} \in (\Omega X) \\ Set_{\mathbb{K}^{\circ\prime}} & \mathbb{K}^{\circ} \notin q_{\ell} \in (\Omega X) \end{cases}$$

and where $\Omega = H_0(G\text{-CW})$ is simply connected.

2.1 Change of basis

Let $\Sigma_{g,n}$ be a based space with crossing number n and genus g, and let there be a bundle $E : \Sigma \longmapsto \mathfrak{b}$, which lifts fiber spectra to a motive over some scheme, $X_b \longmapsto Mot_{\mathfrak{sch}}$. Let m_0 be a marked point of $\Sigma_{g,n}$. The ideal goal is to have a complete geometric interpretation of the flow around m_0.

To do so, we substitute the base space (indexed by x) of a Cartesian square by a time-evolved dissipative system. We do so by forcing a metric (in the canonical sense) on the underlying stack of X, giving us the following H-space:

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "\theta"] & \mathfrak{X}' \arrow[d, "\mu"] \\ X \arrow[r, "pr_0"] \arrow[d, "\mu\circ\theta"'] & X' \arrow[d, "\theta\circ\mu"] \\ \Omega X'_{\mu} \arrow[r, squiggly] & \Omega X_{\mu} \end{tikzcd}$$

The projection $X \xrightarrow{pr_0} X'$ is faithful and full. Actually, this space is a heart; in fact, it is $X^{\heartsuit}_{b_q}$.

Definition 2.1. *We call here a "stabilizer" what Bachmann [5] calls the image of a "motivic Tambara functor." We call a $SH^{\mathbb{A}^1}$-realization of this image a "stable realization" at $\mathbb{A}^1$.*

We have:

$$\prod_{\mathfrak{x}^0}^{\mathfrak{x}^1} : 0 \longrightarrow ... \longrightarrow \mathfrak{X} \longrightarrow \mathfrak{X}' \longrightarrow \mathfrak{X}'' \longrightarrow ... \longrightarrow 1$$

such that

$$\pi_0(PShv(X)) = \Pi_{\mathfrak{X}}$$

where

$$[0,1] \subset \mathbb{X}^{\circ}_{Newt}$$

is an edge of a Newton polygon.

The stabilizers in stringy motivic spectral theory are Quillen equivalent to the third element in a triangle containing $GProj(A)$ and $GInj(A)$, where A is an abelian category.[3]

2.2 Koszul duality

Write $\Lambda^{\$}(X)$ once and for all to mean the Koszul dual of X.

We will be discussing Koszul duality on an $\mathcal{A}d\mathcal{S}_5$ brane. Let $\mathscr{O}_{white}$ be a white hole. The Koszul dual

$$\Lambda^{\$}(X)(\mathscr{O}_{white}) = \mathscr{O}_{black}$$

Recall that, from a white hole, any information may be radiated outwards, but that no information may enter. In this way, it is the opposite of a black hole. The point of view I'm taking, is that in $\mathcal{A}d\mathcal{S}_5$, three dimensions are conserved for white holes and two for black holes, giving us a 2/3 spin for the gravitino.

There is an equivalence between dgas in $\Lambda^{\$}(X)$, and cofibrant objects in $\mathcal{B}^{p,q}_{\mathcal{A}d\mathcal{S}_5}$. This is technically an equivalence of Waldhausen categories:

$$\begin{tikzcd} X \arrow[r, "cof"'] \arrow[d] & X' \arrow[d] \\ X'' \arrow[r, "cof"] & X' \cup_X X'' \end{tikzcd} \qquad \simeq \qquad \begin{tikzcd} Y \arrow[r, "cof"'] \arrow[d] & Y' \arrow[d] \\ Y'' \arrow[r] & Y' \cup_Y Y'' \end{tikzcd}$$

where the map $X' \cup_X X'' \longmapsto Y' \cup_Y Y''$ is the diagonal of the brane. Letting each side of the above isomorphism be two dimensional gives us a hyperkaehler manifold $M^{hyp}_{X,Y}$, where X, Y means $fib((X', X''), (Y', Y'')) \cong fib(\Delta_\alpha, \Delta_\beta)$.

[3]See [6]

3 Work Locking and Holes

A work lock acting on a parton in 3-space gives us a 2-dimensional black hole, $\mathscr{O}^2_{black}$. This gives us a globally stable, but locally unstable pure state.

Definition 3.1. *A work lock, W_L, is an obstruction to the long exact sequence:*

$$... \longrightarrow E_{\mathscr{O}^1_{black}} \longrightarrow ..$$

which prevents us from promoting it to the solution of a Klein-Gordon equation.

A work-locked object only exhibits local interactions. This is why, for instance, we have two equations to calculate air resistance at different velocities, one small and one large. Take the space of a non-singular orbifold for which the map

$$\mathcal{O} \longmapsto Man$$

fails to be effective. The span of the orbifold is reduced to

$$o \longleftarrow \mathcal{O} \longrightarrow o + k$$

which forces the effective equivalence of $k \cong o$, so that k belongs to the sheaf $\mathcal{O}_X$.

Let $\Sigma_{g,n}$ be a manifold with genus g and crossing number n. Let this space be generated by $\rho_{mn} = \langle \phi_i(m) | \phi_j(n) \rangle$. Let there be a collection of transition maps $\psi_\Sigma : i \longrightarrow j$, such that $\psi \circ \phi_{k=i,j}$ is a quasi-fibration for all k. The truncation

$$\Sigma_{g,n=\tau \leq k}$$

yields us the effective wall-crossing fuctor

$$\mathfrak{Walls}_{Eff} = \tau \longrightarrow \tau > k$$

which describes the black-white transition. This lifts a (possibly wild) harmonic bundle over a brane to the $\mathfrak{sl}_2 \times SU(2)$ frame.

$$\begin{array}{c} ... \\ \uparrow \\ \mathcal{B}^{d+1}(\Pi_i(p)) \\ \uparrow \\ \mathcal{B}^{d}(\Pi_i(p)) \\ \uparrow \\ \mathcal{B}^{d-1}(\Pi_i(p)) \\ \uparrow \\ ... \end{array}$$

This diagram is commutative up to all coherent isomorphisms. For every object in the above category, there is a Galois connection

$$Gal_p : \Pi_i(p) \longmapsto \mathcal{B}^\bullet$$

which truncates the Ω-spectrum

$$\Omega\mathcal{B}^\bullet_{\tau<k} \longmapsto \Omega\mathcal{B}^\bullet_{\tau\sim k}$$

This "truncation" is, physically speaking, the reversal of a work-lock on a hole, or in other words, the transitions

$$\mathscr{T} : \mathscr{O}_{white} \leftrightarrows \mathscr{O}_{black}$$

Here, $\mathscr{T}$ stands both for truncation and transition. This is an inclusion

$$* \in \mathscr{P}\mathscr{T}$$

where $*$ is a critical point, and where $\mathscr{P}\mathscr{T}$ is our potential theory. It is understood here that the correct way of thinking about this point is as a topological hole, $B_3(\mathscr{O})$; it has a third Betti number of 1.

3.1 Cor_n functors

Recall the correlation given by $Cor_n(q_\ell, \Omega)$:

$$q_\ell^n \propto \Omega^n$$

Let $\mathcal{S}(Cor_n(*))$ denote the category of sheaves: $* \propto \Omega*$ with an action $\tilde{g} : * \longmapsto \mathcal{S}(Cor_n(*))$. The category $\mathcal{S}(Cor_n(*))$ is evidently the target of a Galois functor:

$$Gal_p : \Pi_i(p) \longmapsto \mathcal{S}(Cor_n(*))$$

(for p a good prime), which has the right-lifting property against all $(\alpha \cap \beta) \in Gal_p$. We have, for all $(\alpha, \beta, \gamma) \in Gal_p$:

$$(\alpha \cap \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma) = \alpha \cap \beta \cap \gamma$$

so that the relationship $\mathcal{R}$ given by $\alpha\mathcal{R}\gamma : \alpha \longrightarrow \beta \longrightarrow \gamma \longrightarrow ...$ becomes transitive, and obeys the generalized cocycle condition

$$\alpha_i \cap \beta_j \cap \gamma_k \cong i \longmapsto k$$

A functor $Cor_n(\bullet, \bullet')$ is effectively a Galois connection $Gal_n : \bullet \longmapsto \bullet'$ which sends every closed set in $\bullet$ to an open in $\bullet'$. That is to say, if there is a connective section (say, a triangle) in the preimage, it will be preserved by the projection.

Suppose we are given the functor $Cor_n(q_\ell, \Omega)$. Let q_ℓ be given by a set of paths Σ_ℓ. Then, we obtain a delooping $\Omega^{n-1}q$ for every point $l \in \ell$. This is the theme of Cor_n

A References

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